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Moduli Spaces

Problem session 12/7/2012

Exercise 1. Let us fix a complex torus $T = \mathbb{C}^g/\Lambda$, where Λ is a full-rank lattice. Let $e_1, \ldots, e_g$ be the canonical basis for $\mathbb{C}^g$, and $\lambda_1, \ldots, \lambda_{2g}$ a basis for Λ. Write Π for the $g \times 2g$ matrix whose columns are the λ_js. In this exercise, we see that the torus T is an abelian variety precisely when there exists an integer-valued, non-degenerate alternating matrix A such that

$$\Pi A^{-1} \Pi^t = 0, \quad \Pi A^{-1} \overline{\Pi} > 0 \tag{1}$$

(the latter condition means that the resulting $g \times g$ matrix is positive-definite). The two equations (1) are called *Riemann bilinear relations*.
We have already seen in the lectures that it is possible to identify $H^2(T, \mathbb{Z})$ with the abelian group of integer-valued, alternating bilinear forms on Λ, and that moreover:

- One such form is Hermitian iff the corresponding cohomology class is in $H^{1,1}(T)$ iff the latter is the first Chern class of a line bundle,
- In the previous case, the Hermitian form is positive precisely when the line bundle is ample.

1. Let E be an integer-valued, non-degenerate alternating form on Λ. Extend E to $\mathbb{C}^g$ by posing

$$H(u, v) := E(iu, v) + iE(u, v).$$

Prove that H is Hermitian iff $E(iu, iv) = E(u, v)$ iff the first equation of (1) is satisfied (where A is the matrix representing E).

2. Now given that H is Hermitian, prove that it is positive iff the second condition of (1) is satisfied.

3. Now that we have proven the Riemann bilinear relations, prove the following. The torus $T := \mathbb{C}^g/\mathbb{Z}^g \oplus \Omega\mathbb{Z}^g$ is a principally polarized abelian variety if and only if the matrix Ω is symmetric and positive definite.

4. Let $g = 2$ and consider the torus given by the lattice generated by the vectors $(1,0), (0,1), (i,0), (\sqrt{2}, i)$. Why is this torus not an abelian variety? Can you do other similar examples?

Exercise 2. In this exercise we check that the fact that the action defined in the lectures of the symplectic group on the Siegel upper half space is well-posed. Consider the Siegel upper half space

$$\mathbb{H}_g := \{\tau \in Mat(n,n) |\ \tau = \tau^t, Im(\tau) > 0\},$$

with the action of $Sp(2g, \mathbb{Z})$

$$g\tau := (A\tau + B)(C\tau + D)^{-1}$$

as seen in the lectures.

1. Prove that $C\tau + D$ is invertible.

2. Prove that $g\tau \in \mathbb{H}_g$.

3. Prove that $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$.