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# Moduli Spaces

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**Exercise 1.** In this exercise we study an example of stable reduction. We study the case of the family of elliptic curves

$$C_t : \quad y^2 = x^3 - t$$

with parameter $t \in \mathbb{A}^1$. All fibers for varying $t$ are smooth elliptic curves isomorphic to a fixed elliptic curve $E$. By using the statement of the stable reduction theorem we know *a priori* that the stable reduction process must produce a curve isomorphic to $E$ on the fiber $t = 0$.

1. Blow-up three times the point $(x, y) = (0, 0)$, to obtain a central fiber with four components with normal crossings:

$$D = C + 2E + 3F + 6G,$$

   where $C$ is the proper transform of the original curve $C_0$, $E, F, G$ are smooth genus 0 curves, and the coefficient in front of them is their multiplicities as divisors in the surface. Show that $E \cdot F = E \cdot C = D \cdot C = 0$ and $G \cdot E = G \cdot G = G \cdot E = 1$.

2. * Perform suitable cyclic base changes of order 2 and 3, each followed by a normalization. By cyclic base change of order $p$ we mean a map $t \to t^p$. (Hint: show that the effect on a divisor $D$ of doing base change of prime order $p$ followed by normalization, is to take the cover branched over 0, and reduce the result mod $p$).

3. If $\tilde{C}, \tilde{E}, \tilde{F}, \tilde{G}$ are the fibers over $C, E, F, G$ respectively, then $\tilde{C}$ is a smooth rational curve, $\tilde{E}$ is the disjoint union of two smooth rational curves, $\tilde{F}$ is the disjoint union of three smooth rational curves, and $\tilde{G}$ is smooth curve of genus 1.

4. Show that $\tilde{G}$ is isomorphic to any of the $C_t$'s, for $t \neq 0$.

5. Show that all the rational curves in point (4) have self-intersection $-1$. Indeed the self-intersection of the whole special fiber is 0, and each of those rational curves meets the rest of the special fiber exactly once.

6. Conclude that the minimal model of the resulting surface, obtained by contracting all $(-1)$-curves, has central fiber isomorphic to $\tilde{G}$, hence to any of the $C_t$'s.

7. Show that the same analysis can be performed in the case of a general pencil $C_t$ of curves on a surface, with $C_t$ smooth for $t \neq 0$, and $C_0$ a curve with one cusp that is not a base point of the pencil. Everything will be similar, with the only difference that $\tilde{C}$ is now the normalization of $C_0$. So the central fiber at the final stage of the stable reduction is $\tilde{C}$ glued in the fiber over the cusp at the elliptic curve $\tilde{G}$.