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Moduli Spaces

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Exercise 1. This is a collection of combinatorial exercises on stable curves. Let C be a stable curve of genus $g \geq 2$, with γ irreducible components C_i and δ nodes (double points) P_j.

1. Prove that $\delta \leq 3g-3$ and $\gamma \leq 2g-2$ (Hint: can you compute the first Betti number of a graph in terms of vertices and edges?).
2. Prove that, if C is irreducible, then the maximum number of nodes that C can have is $\delta \leq g$.
3. The curve C is of *compact type* when its dual graph is a tree, or equivalently when $h^1(\Gamma(C)) = 0$. Prove that if C is of compact type, then $\delta = \gamma - 1 \leq 2g-3$.
4. Let $\delta = 3g-3$ and $\nu_i : C_i^\nu \to C_i$ be the normalization of the irreducible component C_i. Prove that, under this hypothesis, $C_i^\nu \cong \mathbb{P}^1$ and $\#\nu_i^{-1}(C_i \cap \operatorname{sing} C) = 3$.
5. If $\delta = 3g-4$, then there exists one j such that, for all $i \neq j$, $C_i^\nu \cong \mathbb{P}^1$ and $\#\nu_i^{-1}(C_i \cap \operatorname{sing} C) = 3$. Moreover, either $C_j^\nu \cong \mathbb{P}^1$ and $\#\nu_j^{-1}(C_j \cap \operatorname{sing} C) = 4$ or C_j^ν is an elliptic curve and $\#\nu_j^{-1}(C_j \cap \operatorname{sing} C) = 1$.
6. Can you provide a recipe to produce, for all g, stable curves with $3g-3$ irreducible components?

Exercise 2. In this exercise we study the moduli space M_3 of smooth, genus 3 curves. Let C be a smooth curve of genus g and K_C its canonical divisor, and

$$|K_C| = \{D \in \mathrm{Div}(C) | \ D \geq 0, D \equiv K_C\}$$

the canonical linear system. Recall that there is always a rational map

$$\phi_{K_C} : C \dashrightarrow \mathbb{P}^{g-1}.$$

Prove that:

1. If $g \geq 2$, the linear system K_C is basepoint free. This means that for all points $P \in C$, one has $h^0(C, K_C - P) = h^0(C, K_C) - 1$. Conclude that ϕ is a morphism (it is defined everywhere).

2. The morphism ϕ is either an embedding, or C is hyperelliptic. What is the degree of $\phi(C)$ in $\mathbb{P}^{g-1}$? Recall that ϕ is an embedding precisely when, for all $P, Q \in C$, one has

$$h^0(C, K_C - P - Q) = h^0(C, K_C) - 2$$

   (K_C is very ample). If this is not so, consider the morphism $\phi_{|P+Q|}$ given by the divisor $P + Q$.

3. Compute the dimension of the hyperelliptic locus

$$H_g := \{[C] \in M_g | \ C \text{ is hyperelliptic}\} \subset M_g.$$

   Hint: how many degrees of freedom do $2g + 2$ points in $\mathbb{P}^1$ have, up to projective equivalence?

4. Let us define

$$U := \{[f] \in \mathbb{P}(\mathbb{C}[X_0, X_1, X_2]_4) | \ V(f) \subset \mathbb{P}^2 \text{ is smooth}\}.$$

   Why is U open and dense in the projective space $\mathbb{P}(\mathbb{C}[X_0, X_1, X_2]_4)$? Prove that there is a morphism $U \to M_3$ whose complement of the image is the hyperelliptic locus H_3. Conclude that M_3 is unirational. What is the fiber of the map $U \to M_3$ over a point?