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Moduli Spaces

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Exercise 1. Consider the linearized action of $\mathbb{C}^*$ (parameter t) on $\mathbb{P}^n$, given by

$$t(x) := \mathrm{diag}(t^{r_0}, \dots, t^{r_n})x$$

for a given $(r_0, \dots, r_n) \in \mathbb{Z}^{n+1}$. Prove directly, from the definition of stability, that

1. x is *stable* if and only if there are i, j such that $x_i \neq 0 \neq x_j$ and $r_i < 0 < r_j$,
2. x is *semistable* if and only if there are i, j such that $x_i \neq 0 \neq x_j$ and $r_i \leq 0 \leq r_j$.

Exercise 2. Consider the linearized action of $G := SL(n+1)$ on $\mathbb{P}^n$

$$A(x) := Ax.$$

Let $\lambda : \mathbb{C}^* \to G$ be a 1-parameter subgroup. Recall the definition of $\mu(\lambda, x)$ given in the lectures.

1. Prove that $\mu(\lambda, x) = \mu(A^{-1}\lambda A, A^{-1}x)$.
2. A 1-parameter subgroup is in its normalized diagonal form if λ is as in Exercise 1, with $r_0 \geq \ldots \geq r_n$, and $r_0 + \ldots + r_n = 1$. Prove that x is a (semi-)stable point precisely when

$$\mu(\lambda, Ax) > 0 \quad (\geq 0) \qquad \forall A \in G$$

for all 1-parameter subgroups λ of G in normalised diagonal form.

Exercise 3. We want to study the moduli space that parametrizes d unordered points on the projective line, modulo projective equivalence. Let us now fix $G = SL(2)$, and define $V_d := \mathbb{C}[x_0, x_1]_d$, the vector space of degree d polynomials in 2 variables. We define the following action of G on $\mathbb{P}^1$, respectively on $\mathbb{P}(V_d)$:

$$A([x]) := [Ax] \quad \text{and} \quad A([f]) := [f \circ A^{-1}],$$

where $[\cdot]$ denotes the canonical map from a vector space to its projective space.

1. Prove that this gives a well-defined action. Show that the map $\mathbb{P}(V_d) \to \mathbb{P}^1$ that associates to each homogeneous polynomial its zero locus is compatible with the action of G on the domain and codomain (in other words, it is G-equivariant). Observe that G acts on $\mathbb{P}^1$ as Möbius transformations (in other words, the whole set of isomorphisms of the projective line).

2. Consider the 1-parameter subgroup λ_r given by $\text{diag}(t^r, t^{-r})$ for $r \in \mathbb{Z}$. Compute $\mu(\lambda_r, [f])$. How does this subgroup act on $\mathbb{P}^1$, respectively on $\mathbb{P}(V_d)$?

3. Compute $\mu(\lambda_r, [f])$ as a function of (r, f). For which f is μ smaller, equal or greater than 0?

4. Using Exercise 2, conclude that $[f]$ is (semi-)stable, precisely when its zero locus does not contain any point of multiplicity $\geq \frac{d}{2}$ (resp. $> \frac{d}{2}$).

5. Consider the moduli problem of parametrizing d points on the projective line, in such a way that each set-theoretic point appears with multiplicity $< \frac{d}{2}$. Is there a coarse moduli space for this problem? For which d is it compact? In the other cases, can you think of any compactification that admits an interpretation as the coarse moduli space of some problem of parametrizing d points on the projective line?