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Moduli Spaces

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Exercise 1. In this exercise we study the classification problem for linear endomorphisms of vector spaces. Namely, we classify up to isomorphism pairs (V, T) where V is a finite dimensional vector space and T is a linear endomorphism of V. This corresponds to classifying $n \times n$ matrices up to similarity.

1. Define the moduli functor End_n. Families over S are pairs (E, T), where E is a rank n vector bundle on S and T is a homomorphism $E \to E$. Define an appropriate notion of isomorphism of families over S.

2. (This step is needed for the following point) Let X be a projective variety, V a trivial vector bundle and L a line bundle on X. Show that if $V \otimes L$ is trivial, then L is trivial.

3. Show that there is no fine moduli space for End_n. (Hint: show that over $\mathbb{P}^1$ there are two non-isomorphic families of endomorphisms that are fiberwise isomorphic).

4. Show that if $n > 1$, there is no coarse moduli space for End_n. (Hint: show that two matrices with the same characteristic polynomial must map to the same point in the coarse moduli space).

5. Consider now the moduli functor End_n^*, parametrizing pairs (E, T) where T_s has n distinct eigenvalues for all $s \in S$ as in point (1). Show that the complement of Δ in $\mathbb{C}^n$ is a coarse moduli space for End_n^*. Here Δ

is the discriminant locus of those points $(a_1, \dots a_n) \in \mathbb{C}^n$ such that the polynomial

$$x^n + a_1 x^{n-1} + \dots + a_n$$

has precisely n distinct roots.

6. * Let End_n^d be the functor as in point (1) that parametrizes pairs (E, T), where T_s is diagonalizable for all $s \in S$. Then $\mathbb{C}^n$ is a coarse moduli space for End_n^d.

7. Now consider M_n, the vector space of $n \times n$ matrices, on which the linear group GL_n acts by conjugation. If M is a matrix, the coefficients of the characteristic polynomial $\sigma_i(M)$ define an equivariant morphism $\sigma : M_n \to \mathbb{C}^n$. Is $(\mathbb{C}^n, \phi)$ a categorical/good/geometric quotient for the action of GL_n on M_n? Let M_n^* be the open subspace of M_n given by diagonalizable matrices. Is $(\mathbb{C}^n, \phi)$ a categorical/good/geometric quotient for the action of GL_n on M_n^*?

Exercise 2. In this exercise, we prove Noether's theorem: if G is a finite subgroup of GL_n, then the ring of invariants $\mathbb{C}[x_1, \dots, x_n]^G$ is generated by finitely many homogeneous invariants. Recall the Reynolds operator

$$R_G(f) := \frac{1}{|G|} \sum_{g \in G} f(g \circ x).$$

What Noether proved is that the ring of invariants is generated by $\{R_G(x^\beta)\}_{|\beta| \leq |G|}$. Here $\beta = (\beta_1, \dots, \beta_n)$ is a multiindex and $|\beta| := \sum \beta_i$ its length.

1. Show that it suffices to prove that, for each α, $R_G(x^\alpha)$ is a polynomial in $\{R_G(x^\beta)\}_{|\beta| \leq |G|}$.

2. We can write compactly

$$R_G(x^\alpha) = \frac{1}{|G|} \sum_{A \in G} (Ax)^\alpha,$$

where $(Ax)^\alpha := \prod (A_i \cdot x)^{\alpha_i}$ and $A_i \cdot x := \sum a_{ij} x_j$.

3. Introduce variables $u_1, \dots, u_n$ and compute

$$(u_1 A_1 \cdot x + \dots + u_n A_n \cdot x)^k = \sum_{|\alpha| = k} a_\alpha (Ax)^\alpha u^\alpha,$$

where a_α is the multinomial coefficient. Summing over all A in G, we get

$$S_k := \sum_{A\in G}(u_1A_1x + \ldots + u_nA_nx)^k = \sum_{|\alpha|=k} |G|a_\alpha R_G(x^\alpha)u^\alpha$$

4. Prove that S_k can be written as a polynomial of $S_1, \ldots, S_{|G|}$. (Hint: use that the Newton polynomials $\{p_i\}_{1\leq i\leq n\}}$ generate the ring of symmetric polynomials $\mathbb{C}[z_1, \ldots, z_n]^{S_n}$. The Newton polynomials are defined as $p_k(z_1, \ldots, z_n) := (\sum z_i^k)$.)

5. From the previous point, we have $S_k = F(S_1, \ldots, S_{|G|})$ for a certain polynomial F. Now expanding the previous equality, and equating the coefficients of u^α on both sides, we obtain that $R_G(x^\alpha)$ is a polynomial in $\{R_G(x^\beta)\}_{|\beta|\leq|G|}$. This concludes the proof of Noether's theorem by point (1).