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Moduli Spaces

Problem session 18/5/2012

Exercise 1. In this exercise we give an alternative and simpler construction of $\mathbb{A}^{2[n]}$, the Hilbert scheme of points in the affine plane. This can be then used as a local model for the Hilbert scheme of points on smooth surfaces.
Let us consider:

$$\tilde{H} := \{(B_1, B_2, i)|\ B_i \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n),\ i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \text{ satisfying conditions (a) and (b)}\},$$

where condition (a) is that B_1 and B_2 commute; condition (b) is that there exists no subspace S of $\mathbb{C}^n$ such that $B_1(S) \subset S$, $B_2(S) \subset S$, and $\mathrm{Im}(i) \in S$.
On $\tilde{H}$ there is an action of $GL_n(\mathbb{C})$

$$g(B_1, B_2, i) := (gB_1g^{-1}, gB_2g^{-1}, g \circ i) \tag{1}$$

One can define $H := \tilde{H}/GL_n(\mathbb{C})$ as a set (or topological space).

1. Let us consider a point of $\mathbb{A}^{2[n]}$: an ideal $I \subset \mathbb{C}[z_1, z_2]$ such that $\mathbb{C}[z_1, z_2]/I$ is an n-dimensional vector space V. To I one can associate $B_i(I)$, the induced multiplication by z_i on B, and $i(I)(1)$, the projection of the 1 element of $\mathbb{C}[z_1, z_2]$ in the quotient V. Show that the two conditions (a) and (b) are satisfied by $(B_1(I), B_2(I), i(I))$.

2. Conversely, if (B_1, B_2, i) are given as in the definition of $\tilde{H}$, we can define a map $\phi : \mathbb{C}[z_1, z_2] \to \mathbb{C}^n$ by $\phi(f) := f(B_1, B_2)i(1)$, and then define I as $\ker(\phi)$. Show that I is a point of $\mathbb{A}^{2[n]}$ by showing that ϕ is surjective.

3. Show that the two maps defined in the previous points are inverses. Show that the map defined in the second point is constant on the orbits of the action 1 of $GL_n(\mathbb{C})$ over $\tilde{H}$. If one assumes the existence of $\mathbb{A}^{2[n]}$ as a smooth quasiprojective scheme, this identification gives the set (topological space) H the structure of a smooth quasiprojective scheme.

   In the next facultative steps, we will see directly that H is a smooth quasiprojective scheme and that it carries a universal family of 0-dimensional schemes of length n. Thus it gives an alternative construction of $\mathbb{A}^{2[n]}$.

4. * We show that $\tilde{H}$ is smooth with its original structure of algebraic variety . We start by observing that condition (b) is an open condition. Thus it is enough to prove that the rank of the differential of the map

$$(B_1, B_2, i) \to [B_1, B_2] \tag{2}$$

   is constant. To show this, it is enough to show that the cokernel of 2 has constant rank. Show that this cokernel is made of matrices $n \times n$ which commute both with B_1 and with B_2.

   - Show that $A \to A(i(1))$ defines a linear map from the cokernel of the differential of 2 to $\mathbb{C}^n$
   - Show that the set $\{B_1^a B_2^b(i(1))\}_{a,b}$ span all $\mathbb{C}^n$ by condition (2). Use this to define an inverse of the linear map defined in the previous point.

   Conclude that the rank of the cokernel of the differential of 2 is constantly equal to n

5. * Prove that the action of $GL_n(\mathbb{C})$ on $\tilde{H}$ is without fixed points (equivalently, the stabilizer is always trivial).

6. * We will comment in the problem classes on the fact that the previous two points imply that H is smooth.

7. * On H there is a family $\mathcal{H}$ of 0-dimensional subschemes of $\mathbb{A}^2$ of lenght n. Recognize this by looking back at the second point of this exercise. We will comment in the problem classes on the fact that this in fact a universal family.