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Moduli Spaces

Problem session 11/5/2012

Exercise 1. (strong form of the Yoneda lemma). In this exercise we prove that there is a natural bijection $\mathrm{Nat}(\nu_X, F) \to F(X)$.
Let us fix a category $\mathcal{C}$, an object X in $\mathcal{C}$ with its associated functor of points ν_X, and let F be any contravariant functor from $\mathcal{C}$ to sets.

1. Define a function $\alpha : \mathrm{Nat}(\nu_X, F) \to F(X)$ in the following way. Let $\tau : \nu_X \to F$ be a natural transformation. Define $\alpha(\tau)$ as the image via $\tau_X : \nu_X(X) \to F(X)$ of the identity inside $\nu_X(X)$.

2. Conversely, define $\beta : F(X) \to \mathrm{Nat}(\nu_X, F)$. Let us take $\xi \in F(X)$, and define $\beta(\xi)$ as the natural transformation τ, such that $\tau_U : \nu_X(U) \to F(U)$ is defined by $\tau_U(f) := (F(f))(\xi)$. Check that with this definition, τ is indeed a natural transformation of functors.

3. Prove that α and β are inverses of each other.

4. Using the strong form of Yoneda, prove that $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ can be identified with $\mathrm{Nat}(\nu_X, \nu_Y)$.

5. Recall the definition of a moduli functor $\mathcal{M}$, and of coarse moduli space M, and state when a coarse moduli space is fine. Let us consider a base S and a family $\xi \in \mathcal{M}(S)$. Is there a natural way to associate with the family ξ a morphism $S \to M$? When is such a morphism unique?

Exercise 2. (Taken from Orsola Tommasi's 2009 problem classes)
In this exercise we study the Hilbert Scheme of hypersurfaces in $\mathbb{P}^r$. Let $d, r \geq 1$.

1. Let us fix $X \subset \mathbb{P}^r$ an hypersurface of degree d.
   Compute the Hilbert polynomial of X. Show that the Hilbert polynomial uniquely determines the degree d and the dimension $r-1$.

2. Let $\operatorname{Spec} A$ be an affine scheme. What is the definition of a family of hypersurfaces over $\operatorname{Spec} A$? What are the equations defining $\mathcal{X}$ inside $\mathbb{P}^r \times \operatorname{Spec} A$?

3. Let $\mathbb{P}^{\binom{r+d}{r}-1} = \operatorname{Proj}(k[a_{i_0,\dots,i_r} : i_0+\dots+i_r = d])$. Show that each family $\mathcal{X} \to \operatorname{Spec} A$ (defined in point (2)) induces a morphism $\phi:\ \operatorname{Spec} A \to \mathbb{P}^{\binom{r+d}{r}-1}$.

4. Let $\mathcal{U} = V(\sigma) \subset \mathbb{P}^r \times \mathbb{P}^{\binom{r+d}{r}-1}$, with
$$\sigma([X_0,\dots,X_r],[a_{i_0,\dots,i_r}]_{i_0+\dots+i_r=d}) = \sum_{i_0+\dots+i_r=d} a_{i_0,\dots,i_r} X_0^{i_0}\cdots X_r^{i_r}.$$
   For each family of hypersurfaces $\Xi \to \operatorname{Spec} A$ consider the morphism $\phi:\ \operatorname{Spec} A \to \mathbb{P}^{\binom{r+d}{r}-1}$, defined in point (3). Show that:
$$\phi^*\mathcal{U} = \mathcal{X}.$$

5. Conclude that $\mathbb{P}^{\binom{r+d}{r}-1}$ is isomorphic to the Hilbert Scheme of hypersurfaces of degree d in $\mathbb{P}^r$.

Exercise 3. In this exercise we compute the dimension of the moduli space of curves $\mathcal{M}_g$ assuming it exists. This argument can be turned into a rigorous mathematical proof after having constructed the moduli space as a scheme.

1. Fix a genus g curve C, and a line bundle D on C of degree d. Prove that the vector space of sections of $\mathcal{O}(D)$ has dimension $d+1-g$ when $d > 2g-2$. (Hint: Serre duality, Riemann-Roch).

2. Show that a (Zariski) open subset of $H^0(C,\mathcal{O}(D))^{\oplus 2}$ parametrizes maps $f: C \to \mathbb{P}^1$ of degree d. Two such maps f and f' are the same exactly when there exists $\lambda \in \mathbb{C}^*$ such that the corresponding sections (s,t) and (s',t') satisfy $s' = \lambda s$, $t' = \lambda t$. Conclude that the space of maps $f: C \to \mathbb{P}^1$ of degree d is naturally parametrized by an open subset of a projective space of dimension $2d-2g+1$.

3. We have seen in the lectures that the moduli space $\operatorname{Pic}^0(C)$ parametrizing degree 0 line bundles on C has a natural structure of a g-dimensional torus
$$H^0(C,\Omega_C)^\vee / H^1(C,\mathbb{Z}).$$

If p is a point of C, the map $\mathrm{Pic}^0(C) \to \mathrm{Pic}^d(C)$ (the set of line bundles of degree d) given by $\mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(p)$ is a bijection.

4. (This point is just a plausibility argument. It can be made into a precise proof afer showing the existence of a geometric structure on the moduli space of degree d line bundles on curves of genus g). In this point we assume that a moduli space $\mathcal{M}_g$ parametrizing smooth curves C of genus g exists and has dimension $q(g)$ (a number we want to calculate). Similarly, we assume that there is a moduli space $\mathcal{P}ic_{d,g}$ parametrizing couples (C, L), where C is a curve of genus g and L is a line bundle of degree d on C. By the previous point, it is reasonable to believe that the dimension of $\mathcal{P}ic_{d,g}$ is $q(g) + g$.

5. From (3) and (4) conclude that the space of maps $C \to \mathbb{P}^1$ of degree d when C varies has dimension $q(g) + 2d - g + 1$.

6. We can compute the dimension of the space of maps $C \to \mathbb{P}^1$ in another way. Prove that, given $2g+2d-2$ distinct points $b_1, \ldots, b_{2g+2d-2}$ on $\mathbb{P}^1$, there is a finite number of degree d covering maps $C \to \mathbb{P}^1$ branched at the b_i's. (Hint: use Riemann existence theorem for coverings of Riemann surfaces).

7. The moduli space parametrizing $2g + 2d - 2$ distinct points on $\mathbb{P}^1$ is the quotient of the complement of the diagonals $x_i = x_j$ in $(\mathbb{P}^1)^{2g+2d-2}$ by the action of the symmetric group $\mathfrak{S}_{2g+2d-2}$. Henceforth, it has dimension $2g + 2d - 2$.

8. The dimension of the space of degree d maps $C \to \mathbb{P}^1$ can be computed in two ways. Equating them, one obtains the number $q(g)$:

$$q(g) + 2d - g + 1 = 2g + 2d - 2.$$