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Moduli Spaces

Problem session 27.4.2012

Exercise 1. (Taken from Orsola Tommasi's 2009 problem classes)
In this exercise we study the moduli space of quadrics in $\mathbb{P}^N_{\mathbb{C}}$ up to projective transformation. (This is actually equivalent to studying them up to an automorphism of $\mathbb{P}^N$). We fix a natural number $N \geq 1$. The objects we parametrize are hypersurfaces of degree 2 in $\mathbb{P}^N$, for a fixed $N \geq 1$. Such an hypersurface can always be written in the form

$$^T x A x = 0$$

for a given $(N+1)\times(N+1)$ symmetric matrix A. Two quadrics are equivalent if and only if there exists an invertible matrix B such that $A' = {}^T BAB$.

1. The points in the moduli space correspond to equivalence classes of such matrices. What are the equivalence classes?

2. (What are the equivalence classes when the field is not $\mathbb{C}$? What are they when the field is $\mathbb{R}$?)

3. Is there a set Y whose points correspond to the equivalence classes of quadrics in $\mathbb{P}^N$?

4. Formulate the notion of family of quadrics over a base S.

5. Let E_k be the square matrix with the first $k+1$ ones on the diagonal, and zeros otherwise, and A any symmetric matrix. Consider the following family of quadrics over $\mathbb{P}^1$

$$X_{\lambda,\mu} : {}^T x(\lambda E_k + \mu A)x = 0, \quad [\lambda : \mu] \in \mathbb{P}^1.$$

What is the induced map $\mathbb{P}^1 \to Y$? Is it possible to give Y a structure of an algebraic scheme (or variety) such that the induced map is a regular map?

Exercise 2. (Taken from Orsola Tommasi's 2009 problem classes)
In this exercise we study the moduli space of 2-dimensional planes in an n-dimensional ambient space, and we see it is projective.
On the vector space $\mathrm{mat}(2, n)$ ($n \geq 2$) of $2 \times n$ matrices, we consider the equivalence relation given by $GL(2)$-left multiplication:

$$A \sim A' \Leftrightarrow (\exists B \in GL(2) : BA = A').$$

1. Consider the rational map

$$q : \mathrm{mat}(2, n) \dashrightarrow \mathbf{P}^{\binom{n}{2}-1}$$

   given by

$$q \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \end{pmatrix} = \left[\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}, \ldots, \begin{vmatrix} a_{1,n-1} & a_{1,n} \\ a_{2,n-1} & a_{2,n} \end{vmatrix} \right].$$

   What is the domain of definition of q?

2. Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & v_3 & v_4 & \ldots & v_n \\ 0 & 1 & w_3 & w_4 & \ldots & w_n \end{pmatrix}, \tag{*}$$

   where $v_3, \ldots, v_n, w_3, \ldots, w_n$ are given complex numbers. Compute $q(A)$.

3. Let $B \in \mathrm{mat}(2, n)$ be a matrix of rank 2. How many matrices of the form (*) are equivalent to B?

4. Conclude that two matrices A and A' of rank 2 are equivalent precisely when $q(A) = q(A')$.

5. Calculate the dimension of the image of q.

6. What does $\mathrm{mat}(2, n)/\sim$ parametrize?

7. Is it possible to generalize this to matrices in $\mathrm{mat}(k, n)$?

Exercise 3. In this exercise we prove that there is no fine moduli space of elliptic curves.

1. Let $f : \mathbb{A}^1_0 \to Y \subset \mathbb{P}^N$ be any regular morphism from the punctured affine line ($\mathbb{C}^*$ with the structure of an affine algebraic scheme) to any projective scheme Y. Prove that this map extends uniquely to a map $\mathbb{P}^1 \to Y$. Hint: clear denominators.

2. (In fact, it is always the case that a map $C \setminus \{p\} \to Y$ can be extended (uniquely) to a map $C \to Y$ for C a curve and Y a proper (separated) algebraic scheme. This is called valuative criterion of propernes (separatedness)).

3. Prove that any map $\mathbb{A}^1_0 \to C$, where C is a curve of positive genus, must be the constant map. (Hint: use Riemann-Hurwitz formula).

4. Consider the family g of elliptic curves over $t \in \mathbb{A}^1_0$

$$ty^2 = f(x) \subset \mathbb{A}^1_0 \times \mathbb{P}^2$$

where f is any given cubic or quartic polynomial. Show that each section of the family corresponds to a map $\mathbb{A}^1_0 \to E$ (Hint: perform a base change $\mathbb{A}^1_0 \to \mathbb{A}^1_0$ by changing the variable $t = s^2$).

5. Conclude that the sections of g correspond to the zeroes of f.

6. Why is it not possible for g to be the trivial family (*i.e.* isomorphic to $\mathbb{A}^1_0 \times E$)?

7. Show that all fibers in the family g are isomorphic to a unique elliptic curve E. Conclude that there exists no fine moduli space of elliptic curves.