L6

$$\begin{array}{ccc} & \xleftrightarrow{\text{BRST}} & \\ X^\mu, & c, b & \\ & & \Big\updownarrow \text{WS-susy} \\ \psi^\mu, & \beta, \gamma & \end{array}$$

BRST: phys states of a gauge thy
= cohomology classes

Generically:

Gauge algebra: $[K_i, K_j] = f_{ij}{}^k K_k$

(may be finite or infinite dim)

c^i ferm parameter "ghost"
b_i anti-ghost
$\{c^i, b_j\} = \delta^i_j$

$Q_{BRST} = c^i \left(K_i + \frac{1}{2} K_i^{gh.}\right)$
$K_i^{gh} = -f_{ij}{}^k c^j b_k$

Jacobi Id $Q_{BRS}^2 = 0$

$[H, Q_{BRS}] = 0$

Phys states $Q_{BRS}|\phi\rangle = 0$

Pure gauge states (Null)

$\phi = Q_{BRS}|\psi\rangle$

gauge equivalent state

$|\phi\rangle \simeq |\phi'\rangle \Leftrightarrow |\phi\rangle - |\phi'\rangle = Q_{BRS}|\psi\rangle$

Hilbert space

$\mathcal{H} = \mathcal{F}/\simeq$

Bosonic str (X^M, c, b)

$$Q_{BRS} = \sum_m : c_{-m}\left(L_m^X + \frac{1}{2} L_m^{bc}\right):$$

$$c_{-m} = C^m$$

$$Q_{BRS} = \oint \frac{dz}{2\pi i}\, J_{BRS}(z)$$

$$Q^2_{BRS} = \frac{1}{2}\{Q_{BRS}, Q_{BRS}\}$$

$$= \frac{1}{2}\sum_{m,n}\Big([L_m^{tot}, L_n^{tot}]$$

$$-(m-n) L_{m+n}^{tot}\Big) c_{-m} c_{-n}$$

$$L_m^{tot} = L_m^X + L_m^{bc}$$

$$Q^2_{BRS} = 0 \iff D = 26$$

$|0\rangle$ $\quad sl_2\mathbb{C}$ vac

$L_m|0\rangle = 0 \qquad m \geq -1$

$p^\mu|0\rangle = 0 \qquad \alpha^\mu_m|0\rangle = 0 \qquad m > 0$

$b_m|0\rangle \qquad m \geq -1$

$c_m|0\rangle \qquad m \geq 2$

$c_1|0\rangle = c(0)|0\rangle \neq 0$

$c_0 c_1|0\rangle = (c\partial c)(0)|0\rangle \neq 0$

$[L_0, c_1] = -c_1 \qquad [L_0, c_0] = 0$

$|0\rangle \qquad L_0 = 0$

$c_1|0\rangle, c_0c_1|0\rangle \quad L_0 = -1 \quad Q_{gh} = 1, 2$

$\cancel{\| c_1|0\rangle \|^2 = 0 \qquad \| c_0 c_1|0\rangle \|^2 = 0}$

$Q_{BRS}|0\rangle \neq 0$

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$\langle q' | q \rangle \neq 0$ if

$q' = -q - Q \underset{\uparrow}{=} -q + 3$

$Q = -3$

$\langle 0 | [(c_{-1} c_0 c_1) | 0 \rangle] \underset{=1}{\neq 0}$ normalize

$= (\langle 0 | c_{-1} c_0)(c_1 | 0 \rangle)$

Physical ("Tachyon") vacuum

$|0\rangle_T = c_1 |0\rangle$

$\langle 0|_T = (|0\rangle_T)^\dagger = \langle 0 | c_{-1} c_0$

$Q_{BRS} |0\rangle_T = 0$

$c_m |0\rangle_T = 0 \quad m > 0$

$b_m |0\rangle_T = 0 \quad m > 0$

${}_T\langle 0| \cdots |0\rangle_T = \langle 0 | c_1 c_0 c_{-1} \cdots |0\rangle$

Physical states

$$|\Phi\rangle = \underbrace{|\phi\rangle_X}_{X} \otimes \underbrace{|0\rangle_T}_{b,c}$$

$Q_{BRS}|\Phi\rangle = 0 \iff$

$h_\phi = 1$

$Q_{BRS}^2 = 0 \Rightarrow (L_0^X - 1)|\phi\rangle = 0$

$L_n^X |\phi\rangle = 0$

$$T_F(z) = \tfrac{i}{2} :\psi^\mu(z)\,\partial X_\mu(z):$$

$$T_F(z)\,\partial X^\mu(w) = \tfrac{1}{2}\,\frac{\psi^\mu(w)}{(z-w)^2}$$

$$T_F(z)\,\psi^\mu(w) = -\tfrac{1}{2}\,\frac{\partial X^\mu(w)}{z-w}$$

moden $\left\langle \begin{array}{ll} F_m & , m \in \mathbb{Z} \\ G_r & , r \in \mathbb{Z} + \tfrac{1}{2} \end{array} \right.$

$$T(z)\,T_F(w) = \frac{\tfrac{3}{2} T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \dots$$

$$h_{T_F} = \tfrac{3}{2}$$

$$T_F(z)\,T_F(w) = \frac{c_{X,\psi}}{3(z-w)^3} + \frac{2T(w)}{z-w}$$

$$T(z)\;T(w)$$

$N = 1$ SCA $\left\langle \begin{array}{l} NS \\ R \end{array} \right.$

Fermionic sector

$$\psi^\mu(z)\psi^\nu(w) = \frac{\delta^{\mu\nu}}{z-w}$$

$$T_\psi(z) = \frac{1}{2} :\psi(z)\partial\psi(z):$$

$$\cancel{T_F(z) = \frac{i}{2} :\psi^\mu(z)\partial X_\mu(z):}$$

$$\cancel{T(z) = T_X(z) + T_\psi(z)}$$

$$T(z)\psi^\mu(w) = \frac{1/2\,\psi^\mu}{(z-w)^2} + \frac{\partial\psi(w)}{z-w} + \dots$$

$$\Rightarrow h_\psi = 1/2$$

$$T_\psi(z)T_\psi(w) = \frac{D/2}{2(z-w)^4} + \frac{2T_\psi(w)}{(z-w)^2} + \frac{\partial T_\psi(w)}{z-w} + \dots$$

$$c_\psi = D/2$$

$$c_{X,\psi} = c_X + c_\psi = 3/2\, D$$

$$T(z) = T_X(z) + T_\psi(z)$$

Superghosts

		$\lambda, 1-\lambda$	ε	Q	c
diffeom	b, c	$2, -1$	1	-3	-26
susy	β, γ	$3/2, -1/2$	-1	2	11

$$c_{b,c,\beta,\gamma} = -26 + 11 = -15$$

$$c_{X,\psi} + c_{b,c,\beta,\gamma} = 0 \Rightarrow c_{X,\psi} = 15 \Rightarrow D = 10$$

generic b, c

$b_n |0\rangle = 0 \quad n \geq -1 \Leftrightarrow n > -2$

$c_n |0\rangle = 0 \quad n \geq 2 \qquad\qquad \lambda = 2$

$[L_0, c_m] = -m c_m$

$b_n |0\rangle = 0 \quad n > -\lambda$

$c_n |0\rangle = 0 \quad n \geq \lambda$

$0 < n < \lambda$: $\qquad c_n |0\rangle$ have $L_0 = -n < 0$

Fermionic $\quad c_n^2 = 0 \quad \rightsquigarrow$ can fill states

$\rightsquigarrow$ finite min-energy ?

Bosonic $\overbrace{c_n c_n c_n \cdots}^{k} |0\rangle \neq 0$

$0 < n < \lambda$

$L_0 = - n \cdot k$

not bounded below.

$\beta_n |0\rangle = 0 \quad n > -\frac{3}{2}$

$\gamma_n |0\rangle = 0 \quad n \geq \frac{3}{2}$

modes: $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ $\quad L_0 \quad$ R

n: $2, 1, 0, -1, -1$ $\quad$ NS

lower L_0

Bose sea

no transition between "vacua"

∞ number of "q-vacua"

$q \in \mathbb{Z}$ NS $\qquad q \in \mathbb{Z} + \frac{1}{2}$ R

$b_n |q\rangle = 0 \quad n > \epsilon q - \lambda$

$c_n |q\rangle = 0 \quad n \geq -\epsilon q + \lambda$

"(super)ghost pictures"

$\langle q' | q \rangle \neq 0 \quad q' = -q - Q = -q - 2$ (2

$$\langle 0 | \mathcal{O}_{(p)} | 0 \rangle \neq 0$$

$$\Rightarrow p = -2$$

$$\langle -2 | \mathcal{O}_{(p)} | 0 \rangle \neq 0$$

$$\Rightarrow p = 0$$

Vertex operators

WS bosons

$|k\rangle$ $\quad :e^{ik_\mu X^\mu(z)}:$

$\alpha^\mu_{-1}|k\rangle$ $\quad :\partial X^\mu(z)\, e^{ik_\mu X^\mu(z)}:$

$\alpha^\mu_{-m}|k\rangle$ $\quad :\partial^m X^\mu(z)\, e^{ik_\mu X^\mu(z)}:$

WS fermions

NS

use bosonization

$b^{\mu_1}_{-r_1} \cdots |k\rangle$ $\quad :e^{i\vec{v}\cdot\vec{H}(z)}\, e^{ik_\mu X^\mu(z)}:$

canonically appear once (Fermi)

$r_i \in \mathbb{Z}+\frac{1}{2}$ (cyl)

$\vec{H} = (H_1, \ldots H_m)$

$\vec{v} \in D_m^{(0),(v)}$ (plane)

R

$d^{\mu_1}_{-m_1} \cdots |k\rangle$

$m_i \in \mathbb{Z}$ (cyl)

$[L_0, d^{\mu_1}_0] = 0 \qquad \{d^{\mu_1}_0, d^{\mu_2}_0\} = \eta^{\mu_1\mu_2}$

Clifford alg $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$

suggest: $\alpha_0^{\mu} \cong \frac{1}{\sqrt{2}}\gamma^{\mu}$

action on states

$$\begin{array}{ccc} \mathrm{Cliff}(1, D-1) & \supset & \mathrm{Spin}(1, D-1) \\ & & \Big\downarrow {\scriptstyle 2:1} \\ & & SO(1, D-1) \end{array}$$

Representation of Cliff.: <u>irrep</u> gives Spin rep, dim $= 2^{[D/2]}$

D even $\Rightarrow$ dim $= 2^{D/2}$

Explicit construction

Set $\gamma^{\mu} = \sqrt{2}\,\alpha_0^{\mu}$

$$\Psi_I^{\pm}, \quad I = 0, \dots, \tfrac{D-2}{2} \quad \begin{cases} \psi_0^{\pm} = \frac{i}{2}(\gamma^0 \pm \gamma^1) \\ \psi_i^{\pm} = \frac{1}{2}(\gamma^{2i} \pm i\gamma^{2i+1}) & i = 1, \dots, \frac{D-2}{2} \end{cases}$$

$$\{\psi_I^+, \psi_J^-\} = \delta_{IJ}$$

$$\psi_I^+ |0\rangle_R = 0$$

$$(\psi_1^-)^{m_1} \cdots (\psi_n^-)^{m_n} |0\rangle_R$$

$$m_1, \ldots m_n \in \{0, 1\}$$

$$\dim = 2^n = 2^{D/2}$$

Notation $\quad |0\rangle_R = |+\tfrac{1}{2}, \ldots +\tfrac{1}{2}\rangle$

$$\underbrace{|\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \ldots\rangle}$$

$+$ if $m_i = 0$

$-$ if $m_i = 1$

weight in $S+C$

~~rep~~ rep of D_n

$$\alpha_0^\mu |\alpha\rangle = \frac{1}{\sqrt{2}} \gamma^\mu{}_\alpha{}^\beta |\beta\rangle$$

$\alpha \beta = 1, \ldots 2^n$ $\qquad$ spinor indices

$\left((\alpha_0^\mu)^* = \alpha_0^\mu \to \text{Majorana!} \right)$

3

Vertex Operator R-sector

$$b^{\mu_1}_{m_1} \cdots |k\rangle = :e^{i\vec{v}\vec{H}}\, e^{ik_\mu X^\mu}:$$

$$\vec{v} \in D_n^{(s),(c)}$$

$$\cancel{|\ldots\rangle} = |\vec{\theta}\rangle = |\pm\tfrac{1}{2} \ldots \pm\tfrac{1}{2}\rangle$$

$$|\vec{\theta}, k\rangle \qquad \underbrace{:e^{i\vec{\theta}\vec{H}}\, e^{ik_\mu X^\mu}:}_{}$$

$$h = \tfrac{1}{2}\vec{\theta}^2 + \tfrac{1}{2}k^2$$

$$= \tfrac{1}{2}\tfrac{5}{4} + \tfrac{1}{2}k^2$$

Dirac $= 1$

spin-states

$$\Rightarrow M^2 = -k^2 = 2\left(\tfrac{5}{8} - 1\right)$$

$$= -\tfrac{6}{8} < 0$$

Vir constraints $\Rightarrow M^2 = 0$

Missing the ghost contribution.

R-sector → $Q^{\beta,\gamma}_{gh} \in \mathbb{Z} + 1/2$

$|0,k\rangle$ old cov ⇝ $|0,k,q\rangle$ BRS

$Q^{\beta,\gamma}_{gh}$

4-5

Vertex operators (ferm)

$$V = \, e^{i\vec{\lambda}\vec{H} + iq\phi} \, e^{ik_\mu X^\mu} :$$

$\lambda \in D_5^{(s),(c)} \quad q \in \mathbb{Z} + \frac{1}{2}$ R

$\lambda \in D_5^{(0),(v)} \quad q \in \mathbb{Z}$ NS

$$\alpha' M^2 = 4\left(\frac{1}{2}\vec{\lambda}^2 + N - 1 - \frac{1}{2} q(q+2) \right)$$

$N_{gh} = 0$, $N_{sgh} = q$

ghost picture

Define "D_1 - lattice": $D_1 \supset \frac{1}{2}\mathbb{Z}$

Define "$D_{5,1}$ lattice"

$w = (\vec{\lambda}, q) \in D_{5,1}$

$w \cdot w' = \vec{\lambda}\vec{\lambda}' - qq'$

$$V_w \cdot V_{w'} = (z-w)^{\vec{\lambda}\vec{\lambda}' - qq'} \, V_{w+w'} + \ldots$$