Bosonization ⇄ Fermionization

$$\psi^{\pm}(z) = \, : e^{\pm i \phi(z)} :$$

$$: e^{i\phi(z)} : \, : e^{-i\phi(w)} : \; = \frac{1}{z-w} : e^{i(\phi(z)-\phi(w))} :$$

$$= \frac{1}{z-w} : e^{i(\phi(w) + (z-w)\partial\phi(w) + \cdots - \phi(w))} :$$

$$= \frac{1}{z-w} : e^{i(z-w)\partial\phi(w) + \cdots} :$$

$$= \frac{1}{z-w} \left(1 + i(z-w)\partial\phi(w) + \cdots \right)$$

$$= \frac{1}{z-w} + i\partial\phi(w) + O(z-w)$$

$$\psi^{+}(z)\psi^{-}(w) = \frac{1}{z-w} + \, :\psi^{+}(w)\psi^{-}(w): \, + \cdots$$

"B/F Equivalence"

$$\psi^+(z e^{2\pi i}) = \pm \psi(z) \quad \begin{cases} NS \\ R \end{cases}$$

$$= e^{i\phi(z e^{2\pi i})} = :e^{i\phi(z) + 2\pi i}:$$

$$\phi(z e^{2\pi i}) = \phi(z) + 2\pi$$

$$\phi : \mathbb{C} \longrightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$$

$$\text{"}R = 1\text{"}$$

$$\psi^+(z) = :e^{i\phi(z)}: \qquad NS$$

$$\psi^+(z) = :e^{\frac{i}{2}\phi(z)}: \qquad R$$

$R \to \frac{1}{2}R$

L+R-movers

Part fct:

$$Z^{\text{Boson}, c=1}_{R=1}(\tau, \bar{\tau}) = Z^{\text{Ferm}, c=1}_{\text{Dirac}}(\tau, \bar{\tau})$$

Summation winding sectors

Sum over spin structs.

[Ginsparg]

2n real fermions $\rightleftarrows$ n bosons

$\psi^l, \bar\psi^l \qquad l = 1, \dots, n$

$$\psi^l(z)\,\bar\psi^k(w) = \frac{\delta^{l,k}}{z-w} + :\psi^l\bar\psi^k(w): + \dots$$

$\phi^l(z) \qquad l = 1, \dots, n$

$$:e^{i\phi^l(z)}::e^{i\phi^k(w)}: =$$

$$\frac{\delta^{lk}}{z-w} :e^{i(\phi^l(z)-\phi^k(w))}: =$$

$$\frac{\delta^{lk}}{z-w} :e^{i\partial\phi^k(w)(z-w)+\dots}: =$$

$$\frac{\delta^{lk}}{z-w} + \underline{\underline{\delta^{l,k}}}\, i\,\partial\phi^k(w) + \dots$$

Jordan

$$:\psi^k\bar\psi^l(z): = -:\bar\psi^l(z)\,\psi^k(w):$$

$$:e^{i\partial\phi^k(z)}\,e^{-i\partial\phi^l(z)}: = :e^{-i\partial\phi^l(z)}\,e^{i\partial\phi^k(z)}:$$

Solution

$c_k : k \longrightarrow \pm 1$

$c_k c_l c_k = (-1) c_l c_k \qquad l \neq k$

$$\psi^{\pm k}(z) = e^{\pm i \phi^k(z)} c_k$$

various explicit constructions

Example

$N_l = \oint \frac{dz}{2\pi i} \, i \partial \phi^l(z)$

Number Op for ϕ^k:

$[N_k, e^{im\phi^l}] = m \, \delta_{k,l} \, e^{im\phi^l}$

Then $c_k = (-1)^{N_1 + \dots N_{k-1}}$

is a solution [Simple]

use momentum op: BLT p339

c_k Klein factors

Cocycle factors

4d Spin-Statistics Thm
[Pauli + Lüders]

$s \in \mathbb{Z} \longrightarrow [\cdot,\cdot]$ Bose-Einstein
$s \in \mathbb{Z}+\frac{1}{2} \longrightarrow \{\cdot,\cdot\}$ Fermi-Dirac

Reason: Causality
holds up to "Klein-transf"
Bgd: Cau (A/CR) may allow
projective reps. → Grupph

3d/2d: more general types
of statistics allow
1+2 → Perm ⊂ Braids
1+1 → B/F equivalence

Monomials (NS)

$$\underbrace{:\psi^{\pm i_1}\cdots\psi^{\pm i_n}:}_{\text{(up to) } n\text{-factors}} =: e^{i\vec\lambda\vec\phi}: c_{\vec\lambda}$$

$\vec\lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda_i \in \{0, 1, -1\}$

$:\psi^{\pm i}\psi^{\pm j}:$ "Lorentz" $so(2m)$ current

include monomials (R)

$$:\psi^{i_1}\cdots\psi^{i_m}: = e^{i\vec\lambda\vec F} c_{\vec\lambda}$$

$\vec\lambda = (\pm 1/2, \dots \pm 1/2)$

Vectors $\vec\lambda$ are in bosonization generate a lattice $\cong \mathbb{Z}^m$, "D_m-lattice" $D_m \simeq so(2m)$

Bosonisation / Fermionisation

$$\psi^{\pm l}(z) \underset{c=n}{\longleftrightarrow} :e^{i\vec{\lambda}\vec{\phi}(z)}: c_{\vec{\lambda}}$$

$l = 1, \dots, n$

$$\begin{pmatrix} \lambda \in D_n^{0,V} & \text{NS Per} \\ \lambda \in D_n^{S,C} & \text{R / Antiper} \end{pmatrix}$$

$X^i \quad i = 1 \dots 16+6 = 22 \quad \longleftrightarrow \quad$ 44 Fermions (R/
22 Fermions (R/

$X^\mu \quad \mu = 0, \dots 3$

$X^i, \psi^i, i = 1, \dots 6 \quad \longleftrightarrow \quad$ 18 Fermions (R/
$c = 9$

$X^\mu, \psi^\mu \quad c = 6$

$\downarrow$

free fermionic construction

$$\left.\begin{matrix} X^i \\ X^\mu \end{matrix}\right\} \text{ 22 Bosons} \qquad c=22$$

$$\left.\begin{matrix} \left.\begin{matrix} X^i\,\psi^i \quad \text{9 Bosons} \\ \psi^{\pm i} = e^{i\vec{\lambda}\vec{\phi}} \\ i=1,\dots 5 \qquad \vec{\lambda}\in D_5 \end{matrix}\right\} c=9 \\ X^\mu,\ \psi^\mu \qquad c=6 \end{matrix}\right\} \quad c=15$$

"free bosonic construct"

covariant lattice construct.

→ also bosonize the (super) ghosts

Lie algebras (simple)

Basis H^i, $i = 1, \ldots l = \mathrm{rk}(g)$ CSA

$E^{\vec{\alpha}}$, $\vec{\alpha} \in \Delta$ ladder operators

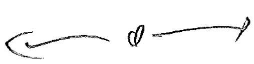
$\Delta = \{\text{roots}\}$

$\dim g = l + |\Delta|$

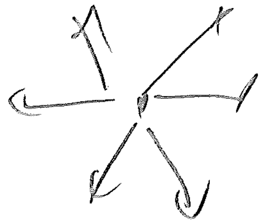
$[H^i, H^j] = 0$

$[H^i, E^{\vec{\alpha}}] = (\vec{\alpha} \cdot \vec{e}_i) E^{\vec{\alpha}}$ $\{e_i\}$ ONB of $\mathbb{R}^l$

$$[E^{\vec{\alpha}}, E^{\vec{\beta}}] = \begin{cases} \varepsilon(\vec{\alpha}, \vec{\beta}) E^{\vec{\alpha}+\vec{\beta}}, & \vec{\alpha}+\vec{\beta} \in \Delta \\ \vec{\alpha} \cdot \vec{H} & \vec{\alpha}+\vec{\beta} = 0 \\ 0 & \text{else} \end{cases}$$



$A_1 \simeq su(2)$



$A_2 \simeq su(3)$

G_2

simply laced, ADE

$A_n \simeq su(n+1)$, $D_n \simeq so(2n)$

E_6, E_7, E_8

ADE

$$[E^{\vec\alpha}, E^{\vec\beta}] = \begin{cases} \varepsilon(\vec\alpha,\vec\beta)\, E^{\vec\alpha+\vec\beta} & \vec\alpha\cdot\vec\beta = -1 \\ \vec\alpha\,\vec H & \vec\alpha\cdot\vec\beta = -2 \\ 0 & \text{else} \end{cases}$$

$$H^i = i\partial X^i$$
$$E^{\vec\alpha}(z) = e^{i\vec\alpha \vec X(z)} = c_{\vec\alpha}$$

$$H^i_m = \oint \frac{dz}{2\pi i} z^m H^i(z)$$
$$E^{\vec\alpha}_m = \oint \frac{dz}{2\pi i} z^m E^{\vec\alpha}(z)$$

$$[H^i_m, H^j_n] = k\, m\, \delta^{ij}\, \delta_{m+n,0} \qquad k=1$$

$$[H^i_m, E^{\vec\alpha}_n] = (\vec\alpha \vec e_i)\, E^{\vec\alpha}_{m+n}$$

$$[E^{\vec\alpha}_m, E^{\vec\beta}_n] = \begin{cases} \varepsilon(\vec\alpha,\vec\beta)\, E^{\vec\alpha+\vec\beta}_{m+n} & \vec\alpha+\vec\beta \in \Delta \\ \vec\alpha\,\vec H_{m+n} + k\, m\, \delta_{m+n,0} & \vec\alpha+\vec\beta = 0 \\ 0 & \end{cases}$$

k is central

$T^a_m \leftrightarrow \{ H^i_m, \vec{E}^{\vec{\alpha}}_m \}$

$[T^a_m, T^b_n] = i f^{ab}{}_c T^c_{m+n}$

$+ k m \delta^{ab} \delta_{m+n}$

$[L_m, T^a_n] = -n T^a_{m+n}$

i.e. $d = -L_0$

$[d, T^a_n] = n T^a_n$

$\langle T^a_m, k, d \rangle = \hat{g}$

untw (affine) KMA

assoc to simple LA g

ext'd chiral alg

$\boxed{\mathrm{Vir} \ltimes \hat{g}}$

$\langle \Delta \rangle_{\mathbb{Z}} = \Gamma_R$ root lattice

$\cong \mathbb{Z}^l$ as abelian group

<u>Rep of g</u>

$$\Phi \left\{ \begin{array}{l} H^i \to \Phi(H^i) \\ E^{\vec\alpha} \to \Phi(E^{\vec\alpha}) \end{array} \right\} \in \operatorname{End} V$$

$$V = \bigoplus_{\vec w \in W(\Phi)} V_{\vec w} \qquad W = \{\text{weights of } \Phi\}$$

$$\Phi(H^i)\, v_{\vec w} = w^i v_{\vec w} \quad \forall v_{\vec w} \in V_{\vec w}$$

$$\Phi(E^{\vec\alpha})\, v_{\vec w} = \begin{cases} 0 & \vec w + \vec\alpha \notin W \\ \in V_{\vec w + \vec\alpha}, & \vec w + \vec\alpha \in W \end{cases}$$

$A_2 \cong su(3)$

adj = (8) || fund = 3 $\qquad \bar{3}$

$T_R = \langle \Delta \rangle_{\mathbb{Z}} = \langle \{ \text{all roots} \} \rangle_{\mathbb{Z}}$

$T_W = \langle \text{weights of all irrep} \rangle$.

$T_R \subset T_W$

$$\left(\Lambda(w)^i = 2 \frac{w \alpha_i}{\alpha_i \alpha_i} \quad \alpha_i \in \Pi \right)$$

~~T_R, T_W are integral lattices~~

~~$T_W \subset T_W^*$~~

ADE: $T_R^* = T_W$

T_R is even and integral

$T^* = \{ v \in \mathbb{R}^l \mid \vec{v} \cdot \vec{\alpha} \in \mathbb{Z} \ \forall \vec{\alpha} \in T \}$

LA Lattice $(g_1, \dots g_n)$ CC

T even $\Leftrightarrow$ $\vec{\alpha} \cdot \vec{\alpha} \in 2\mathbb{Z} \ \forall \vec{\alpha} \in T$

$$(\vec{v}+\vec{\alpha})(\vec{w}+\vec{\beta})$$
$$= \vec{v}\cdot\vec{w} + \underbrace{\vec{v}\cdot\vec{\beta} + \vec{\alpha}\cdot\vec{w} + \vec{\alpha}\cdot\vec{\beta}}_{\in \mathbb{Z}}$$

$$(\vec{v}+\vec{\alpha})\cdot(\vec{v}+\vec{\alpha}) = \vec{v}\cdot\vec{v} + \underbrace{2\vec{v}\vec{\alpha}}_{2\mathbb{Z}} + \underbrace{\vec{\alpha}\cdot\vec{\alpha}}_{2\mathbb{Z}}$$

T_W / T_R finite abelian group

scalar prod mod $\mathbb{Z}$
$\|\cdot\|^2$ mod $2\mathbb{Z}$
} only depends

on T_W / F_R

$D_m^{(V)}$ gen by

$(\pm 1, 0 \ldots 0)$, $(0, \pm 1, 0 \ldots)$

weights of fund repr

$[2m]$, $\mathbb{R}^{2m}$

$D_m^{(0)} = T_R = \{(\lambda_1 \ldots \lambda_m) \mid \lambda_i \in \mathbb{Z}, \sum \lambda_i \in 2\mathbb{Z}\}$

$D_m^{(V)} = T_R + (1, 0 \ldots 0)$

$= \{(\lambda_1 \ldots \lambda_m) \mid \lambda_i \in \mathbb{Z}, \sum \lambda_i \in 2\mathbb{Z}+1\}$

$D_m^{(S)}$ generated by weights of

pos chiral Weyl spinor repr.

$(\pm \frac{1}{2}, \ldots \pm \frac{1}{2})$, even # of $-$

$\dim(S) = 2^{n-1}$

$D_m^{(C)}$ gen by weights of

neg chiral Weyl spinor rep

$(\pm \frac{1}{2} \ldots \pm \frac{1}{2})$ odd # of $-$

$\dim(C) = 2^{n-1}$

The D_n - lattice

$D_n^{(1)}$ gen. by

$(\pm 1, \pm 1, 0 \dots 0), (\pm 1, 0, \pm 1, 0, \dots)$

weights of the adj. repr. of $so(2n)$

Check

$(2 \cdot n)(2 \cdot [n-1])\frac{1}{2} =$

$2n(n-1)$

$\dim so(2n) = \dim \Lambda^2 \mathbb{R}^{2n} =$

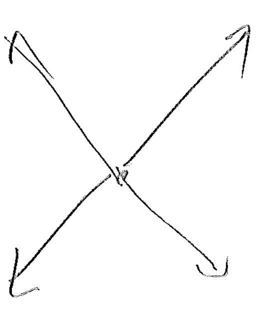
$\frac{1}{2} 2n(2n-1) =$

$\underbrace{\frac{1}{2} 2n(2n-1) - n}_{\#\text{ roots}} \quad + \underbrace{n}_{\text{rank}}$

$\shortparallel$

$n(2n-1) - n = 2n^2 - 2n = 2n(n-1)$

$D_2 \simeq so(4)$



$4+2=6 = \frac{1}{2} 4 \cdot 3$

$D_1 \simeq su(2)$

dim Dirac spinor $= 2^n$ for $SO(2n)$.

$= 2^{n-1} + 2^{n-1}$ ✓

$D_n^{(s)} = T_R + (\tfrac{1}{2} \ldots \tfrac{1}{2})$

$D_n^{(c)} = T_R + (\tfrac{1}{2} \ldots \tfrac{1}{2})$

$T_W = (0) + (V) + (s) + (c)$

grp struct of T_W/T_R

$(0)+(0)=(0)$

$(0)+(v)=(v)$

$(0)+(s)=(s)$

$(0)+(c)=(c)$

$(v)+(v)=(0)$

$(v)+(s)=(c)$

$(v)+(c)=(s)$

$$(s)+(s)=\begin{cases}(0), & n \text{ even}\\ (v), & n \text{ odd}\end{cases}$$

$$(c)+(c)=\begin{cases}(0), & n \text{ even}\\ (v), & n \text{ odd}\end{cases}$$

$$(s)+(c)=\begin{cases}(0), & n \text{ odd}\\ (v), & n \text{ even}\end{cases}$$

$\langle (0),(v)\rangle = \mathbb{Z}_2$

$$\langle (0),(v),(s),(c)\rangle = \begin{cases}\mathbb{Z}_4 & n \text{ odd} \quad \langle s\rangle = \langle c\rangle\\ \mathbb{Z}_2\times\mathbb{Z}_2 & n \text{ even} \quad \langle s, c\rangle\end{cases}$$

$\mathbb{Z}_2^{\text{diag}} = \langle v\rangle$

	(a)	(v)	(s)	(c)
(a)	$2 \simeq 0$	$1 \simeq 0$	0	0
(v)		$1 \simeq 0$	$\frac{1}{2}$	$\frac{1}{2}$
(s)			$\frac{1}{4}n$	$\frac{1}{4}(n-2)$
(c)				$\frac{1}{4}n$

$(1,1,0\ldots 0) \cdot (0,1,1,0\ldots 0)$ $(v \cdot w)$

$(1,1,0\ldots 0) \cdot (1,1,0\ldots 0)$ $(v \cdot v)$

$(1,1,0\ldots 0) \cdot (1,0\ldots 0)$

$n=4$ $\quad 2 \quad \frac{3}{2}$

2

(a) (b) mod $\mathbb{Z}$

(a) (a) mod $2\mathbb{Z}$

Complex

$A < \hat{\Gamma}$, $\Gamma = \hat{\Gamma}/A$

Given (Γ, A), find $\hat{\Gamma}$, A abelian

$$1 \to A \xrightarrow{i} \hat{\Gamma} \underset{s}{\overset{\pi}{\rightleftarrows}} \Gamma \to 1$$

$i(A) \simeq A < \hat{\Gamma}$

$s(\Gamma) \subset \hat{\Gamma}$

~~Dee~~ Determine group structure on $\hat{\Gamma}$

$$s(g) \cdot s(g') = \varepsilon(g,g')\, s(g \cdot g')$$

where $\varepsilon(g,g') \in A$

Assoc:

$$\varepsilon(g,g')\, \varepsilon(gg', g'') = \varepsilon(g', g'')\, \varepsilon(g, g'g'')$$

or $(\delta\varepsilon)(g,g',g'') = e$

A-valued 2-cocycle

$$\varepsilon(g,e) = \varepsilon(e,g) = \varepsilon(e,e) = e$$

two sections define same Γ,
if s, s' differ by a normalized 1-cochain c

$s'(g) = c(g)\, s(g)$

$c(e) = e$

$c \in C^1(G, A)$

$\Gamma \longleftrightarrow (\Gamma, A)$

$H^2(G, A)$

$\Gamma \cong (\mathbb{Z}^n, +)$ $\quad A = (\mathbb{Z}_2, \cdot)$

given commutator:

$s(v)\, s(w) = \delta(v, w)\; s(w)\, s(v)$

$\delta(v, w) = (-1)^{v \cdot w}$

Find cocycle $\varepsilon(v, w)$ st

$\varepsilon(v, w)\, \varepsilon(w, v)^{-1} = \delta(v, w) = (-1)^{vw}$

$A = \mathbb{Z}_2$: $\varepsilon(w,v)^{-1} = \varepsilon(w,v)$

ann!

$\varepsilon(v,w)\,\varepsilon(v+w,x) = \varepsilon(w,x)\,\varepsilon(v,w+x)$

$\varepsilon(v,0) = \varepsilon(0,w) = 1$

Obs: $\varepsilon(v+w,x) = \varepsilon(v,x)\,\varepsilon(w,x)$

$\varepsilon(v,w+x) = \varepsilon(v,w)\,\varepsilon(v,x)$

is gmidda

ADE root lattice

$$\varepsilon(\alpha_i,\alpha_j) = \begin{cases} -1 & \alpha_i\alpha_j = -1,\ i<j \\ +1 & \text{else} \end{cases}$$

$$v * w = \sum_{\substack{i,j \\ j>i}} v^i w^j \alpha_i \alpha_j$$

$\varepsilon(v,w) = (-1)^{v*w}$

$c(v) = (-1)^{\rho * v}$ ρ num of.

on dilbert space

AA: s alle a mit .