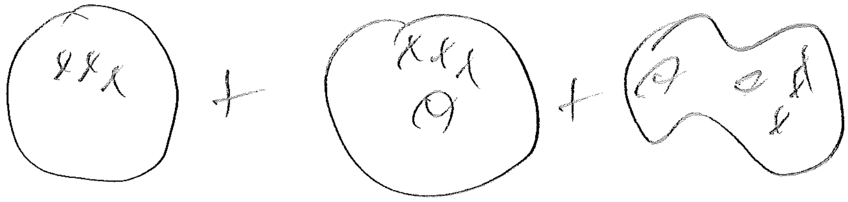
Lecture 3

String interactions

closed, oriented

$A(1,\dots,M) =$  $+ \dots$

$$= \sum_{g=0}^{\infty} \underbrace{\int DX\, Dh\, e^{-S_p[X,h]}\, V_1 \cdots V_M}_{\text{fixed topology}}$$

$M=0$

$$Z = Z_g = \int Dh\, DX\, e^{-S_p[X,h]}$$

$=$  $+ \dots$

fixed topology

no external states

closed, oriented strings/surfaces

gauge fixing: $h = e^{2\phi}\hat{h}$

locally $\hat{h} = \begin{cases} \eta & \text{Lor} \\ \delta & \text{Eu} \end{cases}$

Consider PI for fixed metric

$$\int DX\, e^{-S_P[X,h]}$$

Gaussian integrals

- $$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\mathrm{Re}(a) > 0$$

- $A \in GL(n,\mathbb{R})$ $\quad A = A^T$,

$$\int d^n x\, e^{-\frac{1}{2}x^T A x + bx} = \sqrt{\frac{(2\pi)^n}{\det A}}\, e^{\frac{1}{2}b^T A^{-1} b}$$

- $$S[\phi] = \frac{1}{2}\int d^n x \sqrt{h}\,(h^{mn}\partial_m\phi\,\partial_n\phi + m^2\phi^2)$$
$$= \frac{1}{2}\int d^n x \sqrt{h}\; \phi(\Delta_h + m^2)\phi \qquad \geq 0$$

$$\Delta_h = \ominus \frac{1}{\sqrt{h}}\partial_m \sqrt{h}\cdot h^{mn}\partial_n$$

$$\int D\phi\, e^{-S[\phi]} = \det{}^{-1/2}(\Delta_{h} + m^2)$$

$m = 0 \rightarrow$ ISMN = two models (work)

ζ- det's

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \mathrm{Re}\, s > 1$$

extends merom to $\mathbb{C}$, simple pole at $s = 1$
otherwise holom

$\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$

applic

- $$\left(\sum_{n=1}^{\infty} n\right)_{\zeta} = \zeta(-1) = -\frac{1}{12}$$

- $$\left(\sum_{n=1}^{\infty} 1\right)_{\zeta} = \zeta(0) = -\frac{1}{2}$$

- $\left(\prod_{n=1}^{\infty} a\right)_r = a^{\left(\sum_{n=1}^{\infty} 1\right)_r} = a^{-\frac{1}{2}}$

- $A = A^\dagger$, $\lambda_1 \dots \lambda_N > 0$

$$\zeta_A(s) = \sum_{n=1}^{N} \lambda_n^{-s} \qquad s \in \mathbb{C}$$

converges for large enough Re s

$$e^{-\zeta_A'(0)} = \det A$$

- A = selfadj non-neg diff operator w/ discrete spectrum $\{\lambda_n\}$ on M, compact

$$\zeta_A(s) = \sum_n{}' \lambda_n^{-s} \qquad s \in \mathbb{C}$$

omitting zero modes

conv. for $\mathrm{Re}\, s > \dfrac{\dim M}{\deg A}$

regular at $s = 0$

$$\det{}'_\zeta A = e^{-\zeta_A'(0)}$$

Compute

$$Z = N \int \mathcal{D}X\, e^{-S_P[X,h]}$$

$$S_P[X,h] = \frac{1}{2} \int_\Sigma d^2z \sqrt{h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

$$= \frac{1}{2} \int_\Sigma d^2z \sqrt{h}\; X^\mu \Delta_h X_\mu$$

$$= \frac{1}{2} (X, \Delta_h X)$$

$$\Delta_h = -\frac{1}{\sqrt{h}} \partial_\alpha \sqrt{h}\, h^{\alpha\beta} \partial_\beta$$

$$\Delta_h \psi_m = \lambda_m \psi_m \qquad (\psi_m, \psi_n) = \delta_{m,n}$$

$$X^\mu = \sum_{m=0}^{\infty} a_m^\mu \psi_m = X_0^\mu + X'^\mu$$

$$a_m^\mu \in \mathbb{R}$$

$$1 = (\psi_0, \psi_0) = \int_\Sigma d^2z \sqrt{h}\, \psi_0^2$$

$$\Rightarrow \psi_0 = \left(\int_\Sigma d^2z \sqrt{h} \right)^{-1/2} = \mathrm{vol}_h(\Sigma)^{-1/2}$$

$$X_0^\mu = a_0^\mu \psi_0$$

$$Z = N \int DX \, e^{-\frac{1}{2}(X, \Delta_1 X)}$$

$$= N \int \prod_{\mu, n} da^\mu_n \, e^{-\frac{1}{2}\sum_{n,\mu} \lambda_n (a^\mu_n)^2}$$

$$= N \left(\int \prod_\mu da^\mu_0\right)\left(\int \prod_\mu \prod_{n\neq 0} da^\mu_n \, e^{-\frac{1}{2}\sum_{\mu, n\neq 0} \lambda_n (a^\mu_n)^2}\right)$$

$$\int \prod_\mu dx^\mu_0 = \mathrm{vol}(M)$$

$$x^\mu_0 = a^\mu_0 \psi_0$$

$$\int \prod_\mu a^\mu_0 = \left(\int \prod_\mu dx^\mu_0\right) \psi_0^{-D}$$

$$= \mathrm{vol}(M) \, \mathrm{vol}_1(\Sigma)^{D/2}$$

$$\int \prod_\mu \prod_{n\neq 0} da^\mu_n \, e^{-\frac{1}{2}\sum_{\mu, n\neq 0} \lambda_n (a^\mu_n)^2}$$

$$= \prod_\mu \prod_{n\neq 0}{}' \lambda_n^{-\frac{1}{2}} = \prod_n{}' \lambda_n^{-D/2}$$

$$= \left(\det{}' \Delta_1\right)^{-D/2}$$

$$Z = N \det M \left(\frac{\det' \Delta_1}{\det_n \Sigma} \right)^{-1/2}$$

Fermionic Gaussian integrals

$\theta, \theta^2 = 0$

$\int d\theta\, \theta = 1 \qquad \int d\theta = 0$

$2n$ variables $\theta_i, \bar\theta_i \quad i = 1, \dots, n$

$$\int d\theta_i\, \theta_j = \delta_{ij} = \int d\bar\theta_i\, \bar\theta_j$$

$$\int d\theta_i = 0 = \int d\bar\theta_i$$

Int. over all variables, sign convention

$$\int D\theta\, D\bar\theta \prod_{i=1}^{n} (\theta_i \bar\theta_i) = 1$$

$$Z = \int D\theta D\bar{\theta} \; e^{-\bar{\theta}^T A \theta} = \det A$$

using $\det A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)}$

~~Dirac spinor, massless, flat Lorentzian metric~~

$$\int D\psi D\bar{\psi} \; e^{-\int d^n x \, \bar{\psi} i \gamma^\mu \partial_\mu \psi}$$

$$= \det(i\gamma^\mu \partial_\mu) = \det{}^{1/2}(-\Box)$$

$$i\gamma^\mu \partial_\mu \, i\gamma^\nu \partial_\nu = -\{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu$$

$$= -\eta^{\mu\nu} \partial_\mu \partial_\nu = -\Box$$

Massless Dirac field, on $\mathbb{R}^n$

$$Z = \int D\psi D\bar\psi \, e^{-\int d^n x \, \bar\psi \, i\partial\!\!\!/ \psi}$$

$$= \det(i\partial\!\!\!/) = (\det \Delta_0)^{1/2}$$

$$i\partial\!\!\!/ \, i\partial\!\!\!/ = -\gamma^\mu \partial_\mu \gamma^\nu \partial_\nu$$

$$= -\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu$$

$$= -\delta^{\mu\nu} \partial_\mu \partial_\nu = \Delta_0$$

Integration over metrics

$$Z = N \int Dh\, DX\, e^{-S_P[X,h]}$$

Sym: Rep & Weyl

gauge fix $\quad h_{\alpha\beta} = e^{2\phi}\, \hat{h}_{\alpha\beta}$

locally $\quad \hat{h}_{\alpha\beta} = \eta_{\alpha\beta} / \delta_{\alpha\beta}$

Factorise integration over metrics

$$\delta h_{\alpha\beta} = \overbrace{-(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)}^{\text{Rep}} + \overbrace{2\Lambda\, h_{\alpha\beta}}^{\text{Weyl}}$$

$$= \underbrace{-(Pv)_{\alpha\beta}}_{\text{traceless}} \quad \underbrace{+\, 2\tilde{\Lambda}\, h_{\alpha\beta}}_{\text{trace}}$$

$P: \quad v_\alpha \to (Pv)_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - \nabla_\gamma v^\gamma h_{\alpha\beta}$

$P^\dagger: \quad t_{\alpha\beta} \to (P^\dagger t)_\alpha = -2\nabla^\beta t_{\alpha\beta}$

$t_{\alpha\beta} = t_{\beta\alpha}$

$h^{\alpha\beta} t_{\alpha\beta} = 0$

$$(v, P^+ t) = (Pv, t)$$

$$Dh = D(Pv)\, D\tilde{\Lambda}$$

$$= Dv\, D\Lambda \underbrace{|\det P|}_{=(\det P P^+)^{1/2}}$$

assume: measure is Weyl invariant on zero modes of P, P^+

$\Updownarrow$

vector flds KT $\xleftrightarrow{1:1}$ STT, def of traceless part of metric

$$Z = \underbrace{N \left(\int Dv\right)\left(\int D\Lambda\right)}_{\doteq 1} \int DX\, |\det P|\, e^{-S[X, e^{2\omega}\hat{g}]}$$

finite dim matrices

$\det M^T = \det M$

$\Rightarrow \det M^\dagger = \det M^* = (\det M)^*$

$\Rightarrow \det M M^\dagger = \det M \det M^\dagger$

$= \det M \, (\det M)^*$

$= |\det M|^2$

$\Rightarrow (\det M M^\dagger)^{1/2} = |\det M|$

Express FP determinant as Gaussian integral over anticommuting fermion fields

$$|\det P| = \int Db\, Dc\, e^{-\frac{1}{2\pi}\int d^2\sigma \sqrt{\hat h}\, \hat h^{\alpha\beta}\, b_{\beta\gamma} \nabla_\alpha c^\gamma}$$

$$= \int Db\, Dc\; e^{-S_{ghost}[b, c, \hat h]}$$

$b_{\alpha\beta} = b_{\beta\alpha}$, $h^{\alpha\beta} b_{\alpha\beta} = 0$ antighost $\Updownarrow$ traceless def. of matrix

c^γ ghost $\Leftrightarrow$ vec

$$Z = \int Db\, Dc\, DX\; e^{-S_p[X, \hat h] - S_{gh}[b, c, \hat h]}$$

$$\boxed{\begin{matrix} X\text{-CFT} \\ c \end{matrix}} + \boxed{\begin{matrix} b,c\text{-CFT} \\ c_{ghost} \end{matrix}}$$

$\uparrow$

needed for quantum gauge invariance

Measure i.g. not Weyl invariant

$$Z = \int D\phi\, Db\, Dc\, DX\, e^{-S_P - S_{gh} - (c+c_{gh}) S_L}$$

$S_L = S_L[\phi; \hat{h}]$ Liouville action

ϕ = Liouville field $\hat{=}$ trace part of metric / conf factor

- critical string

$$\left.\begin{array}{l} c + c_{gh} = 0 \\ c_{gh} = -26 \end{array}\right\} \Rightarrow D = c = 26$$

conf anomaly vanishes

- non crit. Str.

WS CFT acquires addl dof ϕ

$D \neq 26$, complicated dynamics

zero modes

$(P\nu)_{\alpha\beta} = 0 \Rightarrow V_\alpha$ CKV

$\hat{=}$ regions which do not change metric

→ overcounting in PI

→ add gauge fixing

→ separate vol(GC)

$(P^\dagger h)_{\alpha\beta} = 0 \Rightarrow h_{\alpha\beta}$ quadratic diff

traceless metric deform which can not be gen by Weyl trf.

"undercounting", no global gauge fixing $h_{\alpha\beta} = e^{2\varphi} \hat{h}_{\alpha\beta}$

On Σ_g :

$$h_{\alpha\beta} \underset{\text{conf}}{\overset{\sim}{=}} e^{\phi}\, \hat{h}_{\alpha\beta}(m_i)$$

metric of const curvature

M_g = { conf. class metrics on Σ_g }
= { conf. structures on Σ_g }
= { cplx. structures on Σ_g }

$$\dim_{\mathbb{C}} M_g = \begin{cases} 0 & g=0 \\ 1 & g=1 \\ 3g-3 & g>1 \end{cases}$$

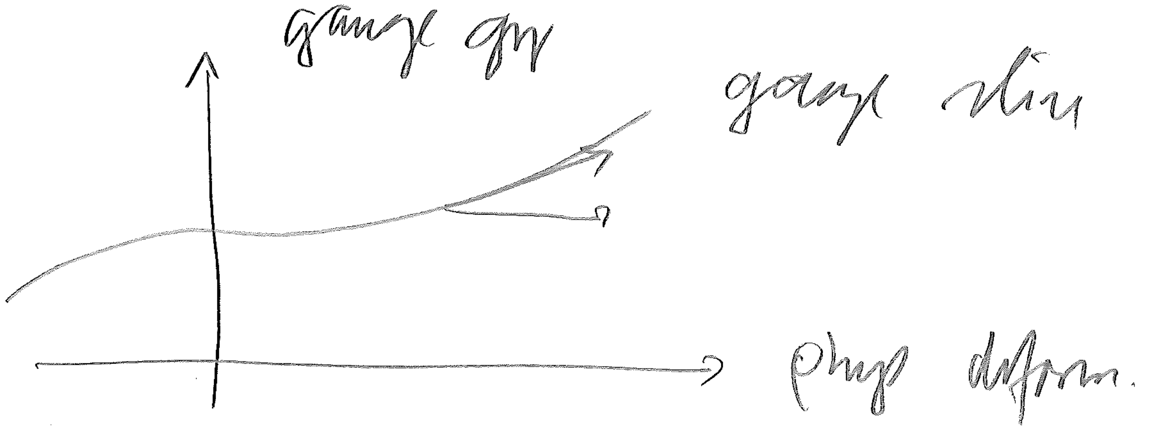
To count all metrics need
to include integration over M_g

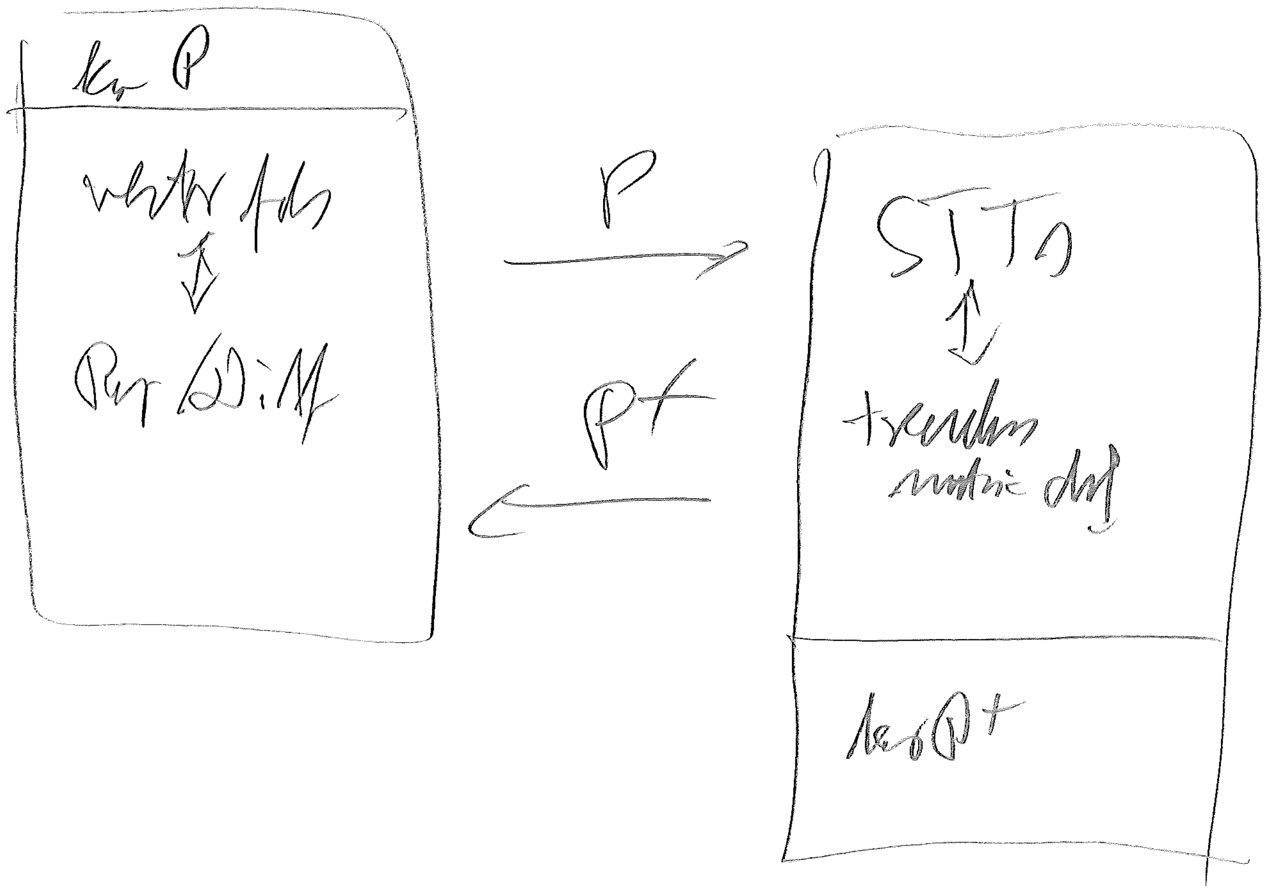
ZI

$$Z = N \; \mathrm{Vol}(BC) \int Db\, Dc\, D\ell \int_{M_g} d\mu(m_i)$$

$$e^{-S_P[X, \hat{h}(m_i)] - S_{gh}[X, \hat{h}(m_i)]}$$

$$\boxed{\begin{array}{ll} CFT & X,\ b,\ c, \\ & m\left(\Sigma_g, \hat{h}(m_i)\right) \\ & \\ & c_{\text{total}} = c_X + c_{gh} = 26 - 26 = 0 \end{array}}$$





$$\mathrm{vol}(CKV)\,\left(\det{}' PP^{\dagger}\right)^{1/2}\, d\mu(m_1 \ldots)$$

A