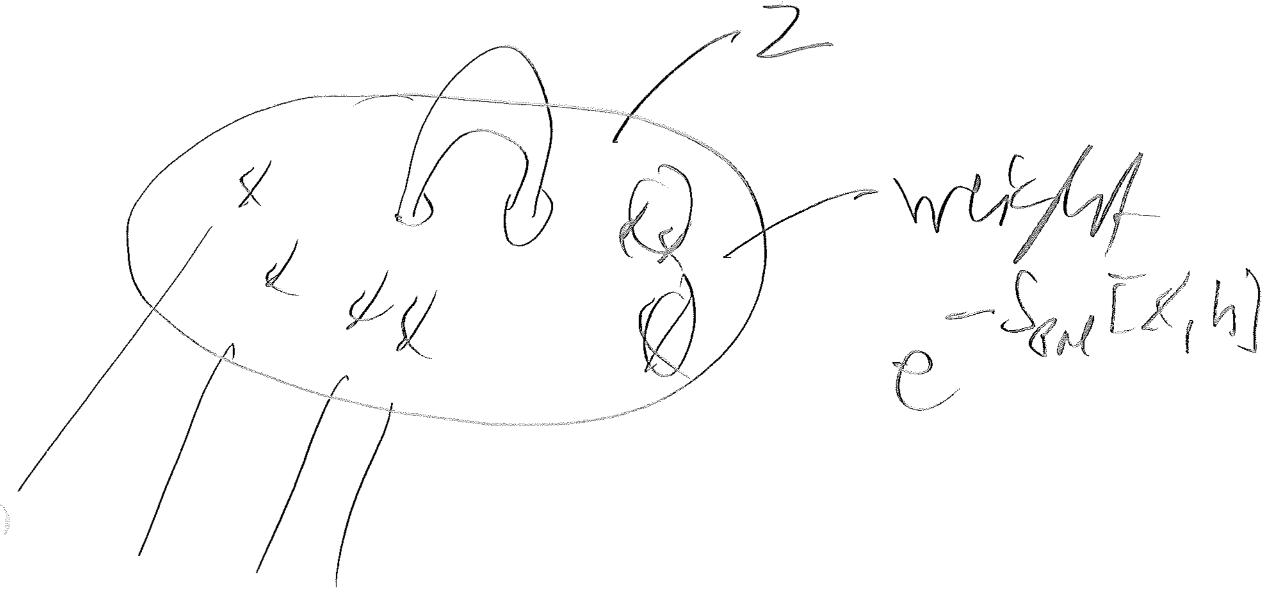
Motivation L2



ext states ≙ vertex operators

$$A(1,\dots,N) = \int DX\, Dh\; e^{-S_P[X,h]}\, V_1 \cdots V_N$$

$$V_i = \int_\Sigma d^2\sigma \sqrt{h}\; \mathcal{O}_i(\sigma)$$

Structures on Σ

Topology ✓

Metric

Conformal Struct

Spin

[illegible]
[illegible]
[illegible]

G-structures

M Mfd

Fiber bdl

$$\begin{tikzcd} F \arrow[r] & B \arrow[d, "\pi"] & \supset & \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi_U"] & U \times F \arrow[dl, "\pi_1"] \\ & M & \supset & U & \end{tikzcd}$$

Vector bdl F = Vector space

G-Principal bdl F = Principal G-[illegible] space

= G-torsor

= free transitive G action

$\pi^{-1}_x M = F_x \cong G$ ([illegible])

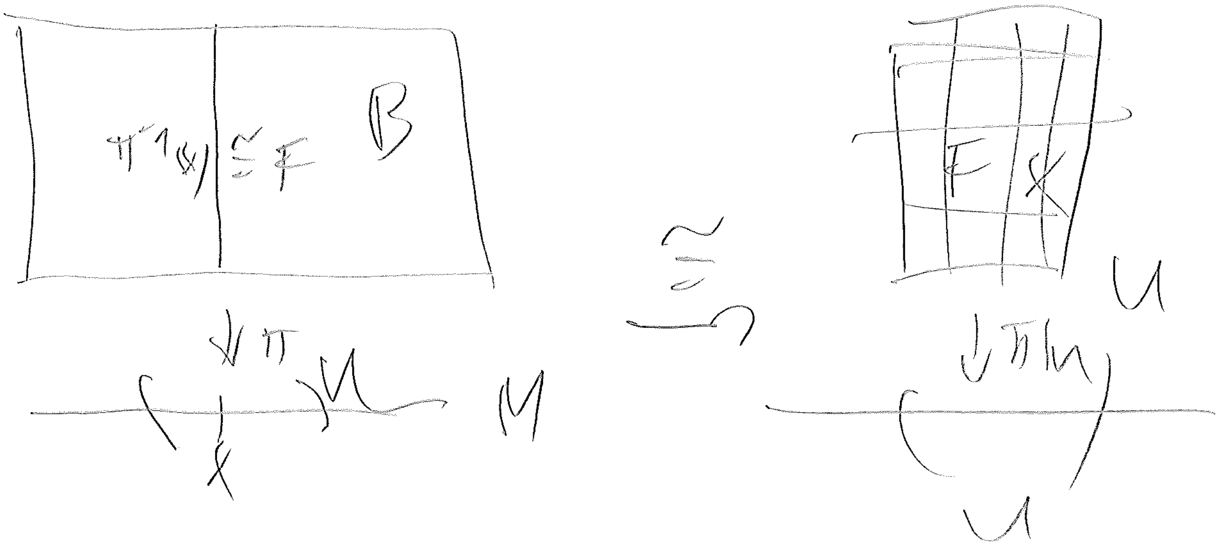
g

like affine bdl.

not gr bdl.

G principal bdl (
repr. of G)

Vector bdl
("same")



- Variant: Individual action of $F \circlearrowleft G$ top grp G

  Variant terms: bundle (w/ G action) / fibre bdl (G action)

- Generalization: Fibration

  local triviality $\to$ homotopy lifting

Fibre bdl w/ grp action

$$\begin{tikzcd} G \circlearrowright F \arrow[r] & B \arrow[d] & U \times F \circlearrowleft G \arrow[d] \\ & M & U \arrow[l, hook'] \end{tikzcd}$$

canonical bdls:

Tg Bdl $\quad TM \quad$ vector bdl $\quad \mathbb{R}^n$

Frame Bdl $\quad FM \quad$ $GL(n,\mathbb{R})$ principal

G struct on M = principal G-subbundle of FM

$G \subset GL(n,\mathbb{R})$

$\exists$ may be obstructed

$\Updownarrow$

∇ w/ Curv + Torsion

prescribed [illegible].

$GL^+(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ orientation

$O(n) \subset GL(n,\mathbb{R})$ Riem metric

not obstructed

$GL(n,\mathbb{R}) = O(n) \cdot Sym(n)$

$\uparrow$ contractible to $\{pt\}$

$O(p,q) \subset GL(n,\mathbb{R})$ pseudoRiem metric

$O(1,1)$ on T, not on Σ_g $g \neq 1$

$SO(n) \subset GL^+(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ oriented Riem.

$GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$ (almost) complex struct

$U(n) \subset SO(2n) \subset GL^+(2n,\mathbb{R}) \subset GL(2n,\mathbb{R})$

$\underbrace{\text{almost Hermitian} \to \ldots \text{Kähler}}_{\text{Torsion of } U(n) \text{ struct}}$

Kähler = 1st order integrable almost Herm.

n-lab

surfaces

$O(2) \subset GL(2,\mathbb{R})$ metric

$SO(2) \subset GL^+(2,\mathbb{R})$ on orientable SF

"$GL(1,\mathbb{C}) \subset GL(2,\mathbb{R})$" conf. struct.

$GL(1,\mathbb{C}) \simeq \mathbb{C}^* \simeq \mathbb{R}^{>0} \cdot U(1) \simeq \mathbb{R}^{>0} \cdot SO(2)$

$GL^+(2,\mathbb{R}) \simeq \mathrm{Sym} \cdot SO(2) \longrightarrow SO(2)$

$\boxed{\text{orientable} \leftrightarrow \text{complex struct.}}$

$GL^+(2,\mathbb{R}) \simeq \mathbb{R}^{>0} \cdot SL(2,\mathbb{R}) \supset SO(2)$

contract

2 components

1 component

closed, orientable SF : complex struct

$=$ $\mathbb{C}$-SF w/o bd, closed $\mathbb{R}$-Surf.

Riemann SF

conn, complex mfd, $\dim_{\mathbb{C}} = 1$

compact — closed w/o bound
— w/ bound ∂ closed

non-compact $\mathbb{C}$, D_{open}

2d real mfd $\longleftrightarrow$ Riemann SF
orientable
(metrizable)

angle struct $\Leftrightarrow$ conf struct
(angle preserving)

$\updownarrow$ $\updownarrow$

cplx mfd / bij holom transf metrics / conf factors

Uniformization thm

Σ simply conn RS $\Rightarrow$

Σ conf equivalent to

- Riem sphere S
- Cplx plane $\mathbb{C}$
- Open unit disk $\cong$ UHP

Σ Riem SF $\Rightarrow$ univ cover

S	elliptic	
$\mathbb{C}$	parabolic	
Disk / UHP	hyperbolic	

closed, oriented 2d Riem mfd $\Rightarrow$

$\exists$ metric conf equival to a Riem metric of const curvature

Metrics of const curv. of universal coverings

R = const.

$S \cong \hat{\mathbb{C}}$ $\qquad ds^2 = \frac{4|dz|^2}{(1+|z|^2)^2}$ $\qquad R>0$

"chordal metric"

$\mathbb{C}$ $\qquad ds^2 = |dz|^2 = dx^2+dy^2$ $\qquad R=0$

$$\left.\begin{aligned} &\text{UHP} \quad ds^2 = \frac{|d\tau|^2}{(\tau_2)^2} \qquad \frac{SL(2,\mathbb{R})}{SO(2)} \\ &\ \updownarrow \\ &\mathbb{D} \qquad \cong \frac{4|dz|^2}{(1-|z|^2)^2} \qquad \frac{SU(1,1)}{U(1)} \end{aligned}\right\} \quad R<0$$

Gauss-Bonnet Thm

$$\boxed{\chi = \underbrace{\frac{1}{4\pi}\int_\Sigma R\sqrt{g}\, d^2x}_{\text{EH action}}} \quad (\text{no bd})$$

closed Σ_g $\qquad \Sigma_0 = \mathbb{C}, \quad \Sigma_1 = T = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

$\Sigma_g = \text{UHP}/\text{Fuchsian grp}$

Fuchsian grp = hyperbolic wallpaper

grp = discrete subgrp of $PSL(2,\mathbb{R})$ w/ compact action. Ex: $PSL(2,\mathbb{Z})$

Const curvature metrics (fixed R)

g	χ	$R(\Sigma_g)$
0	$2>0$	>0
1	0	$=0$
>1	$2-2g<0$	<0

M_g = {cplx strct} $\cong$ {conf strct} $\cong$ {metrics of const curvature}

$$\dim_{\mathbb{C}} M_g = \begin{cases} 0 & g=0 \\ 1 & g=1 \\ 3g-3 & g>1 \end{cases} \leftarrow$$

Riemann-Roch Thm

Description of cplx strcts $\to$
Period matrix $\to$ (Co) homology