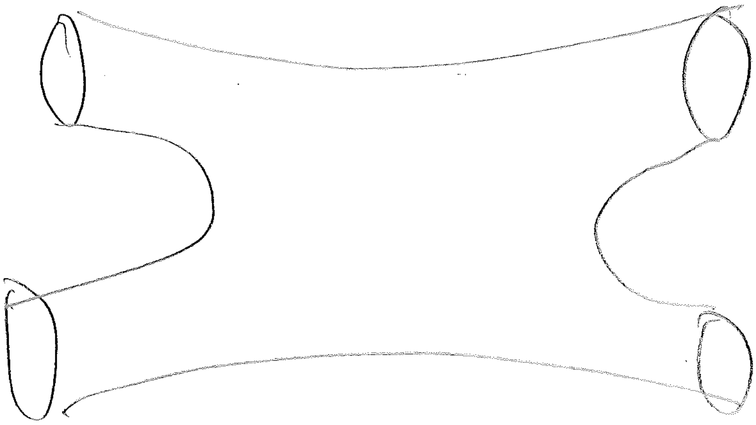
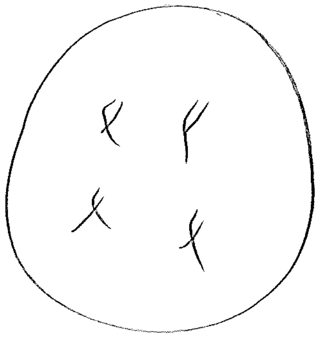
String interactions



$\underset{\text{on shell}}{\cong}$



QFT w/ bd cond
not off-shell

Compact SF w/
marked pts
(V-Op insertions)

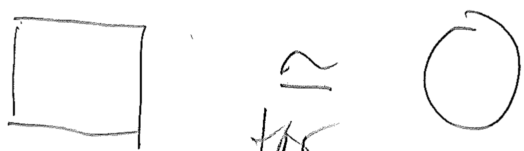
→ String WS = compact SF
(with V Op insertions

Surfaces (Topology)

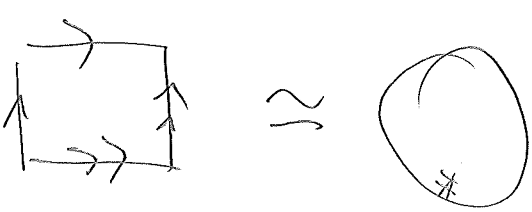
orientable — compact — w/o bd. ("closed")

non-orientable — compact — w/ bd.

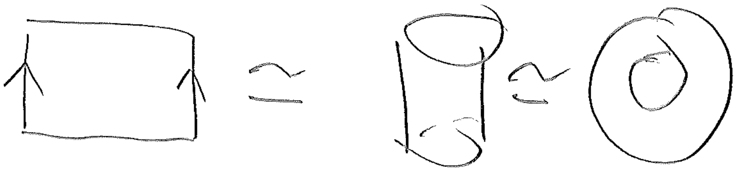
Construct some from polygons



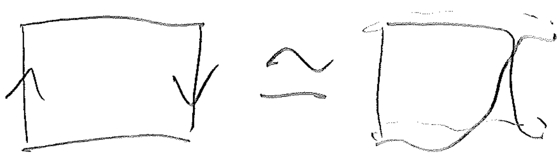
$\simeq$ top (not diff) D (Disk)



$\simeq$ S (Sphere)



$\simeq$ $\simeq$ A annulus cylinder



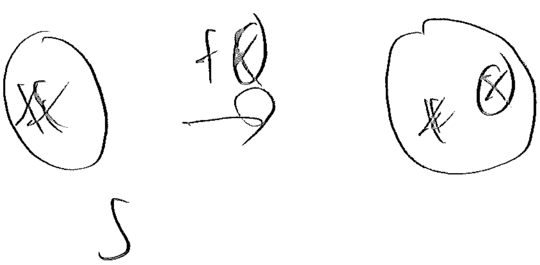
$\simeq$ M Möbius strip

Möbius strip

circle is unknot

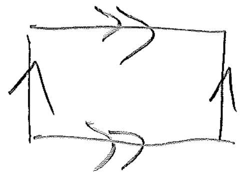
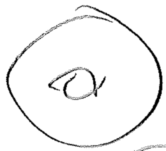
suppose Möbius strip
straight circle no
self-intersection

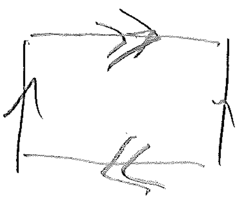
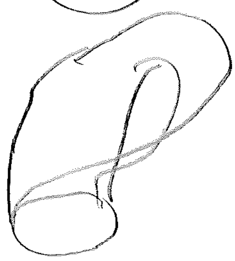
wrong way
straight circle, all of
bend to one side,
self intersection

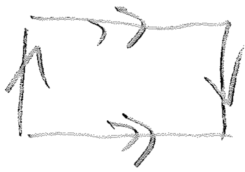


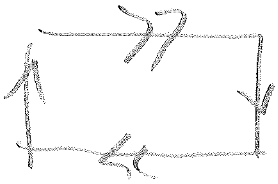
P (Möbius strip glued to disk)

=  ~ $\otimes$ crosscap

 =  T torus

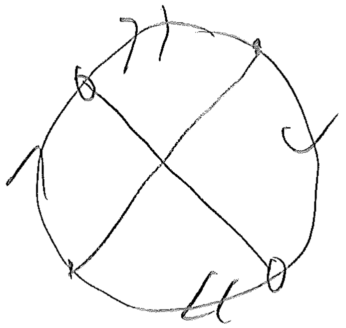
 =  K Klein bottle

 = K

 = P Real Projective Plane

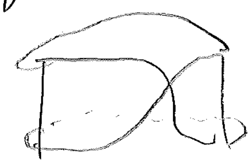
$P = \mathbb{RP}^2$

$\mathbb{R}^2$ / Disk with antipodal pts at ∞ / bd identified



closed,
non orientable

= glue disk on Möbius strip



$\leftarrow$ bd = circle

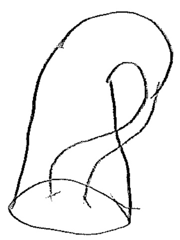
$M \sqcup D$ (disjoint union

$$\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\}) / x \sim \lambda x \quad , \lambda \neq 0$$

$= S^2 / \mathbb{Z}_2$ antipodes identified

$= D^2 / \mathbb{Z}_2$ disk with antipodes on bd identified

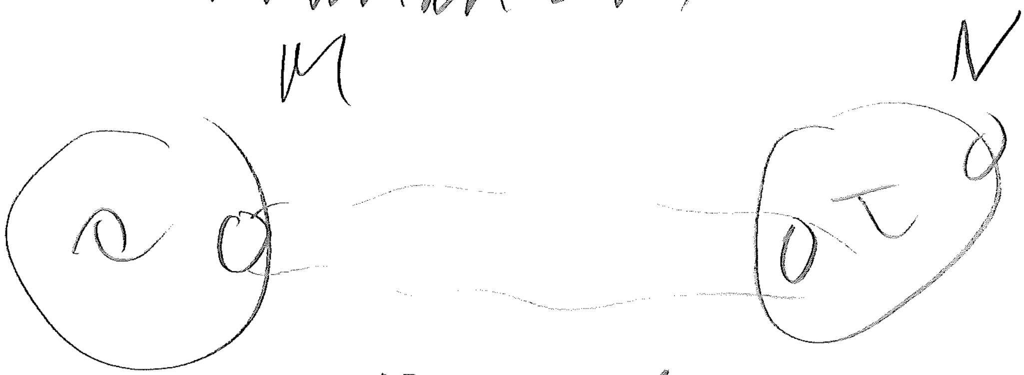
K = Klein bottle "two-torus"



cannot be embedded into $\mathbb{R}^3$ (self-intersection), only immersed

glue 2 Möbius strips

Connected sum



remove disks + glue

$M \# N$

<u>w/o bd</u>

$S \# M = M$ $\qquad$ S = Id

com monoid : # is assoc, with unit.

generated by T, P

$\# T$ add handle

$\# P$ add crosscap

Relation : $P \# P \# P = P \# T$

$P \# P = K$, $P \# K = P \# T$

w/ bd : remove disks.

$X \to X \# P$ add a crosscap

cut disk, glue Möbius

Classification of closed SF (connected)

add handle

$S = \#^0 T$

$T = \#^1 T$

$T \# T$

$\vdots$

$\left.\right\}$ $\#^n T$, $n \geq 0$ orientable

add ⊗

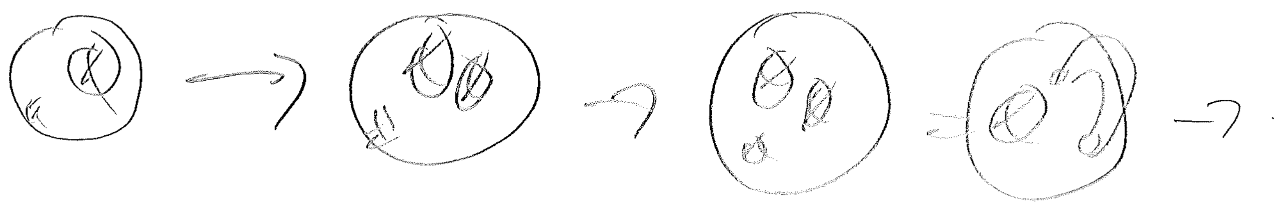
add ⊗

$P = \#^1 P$

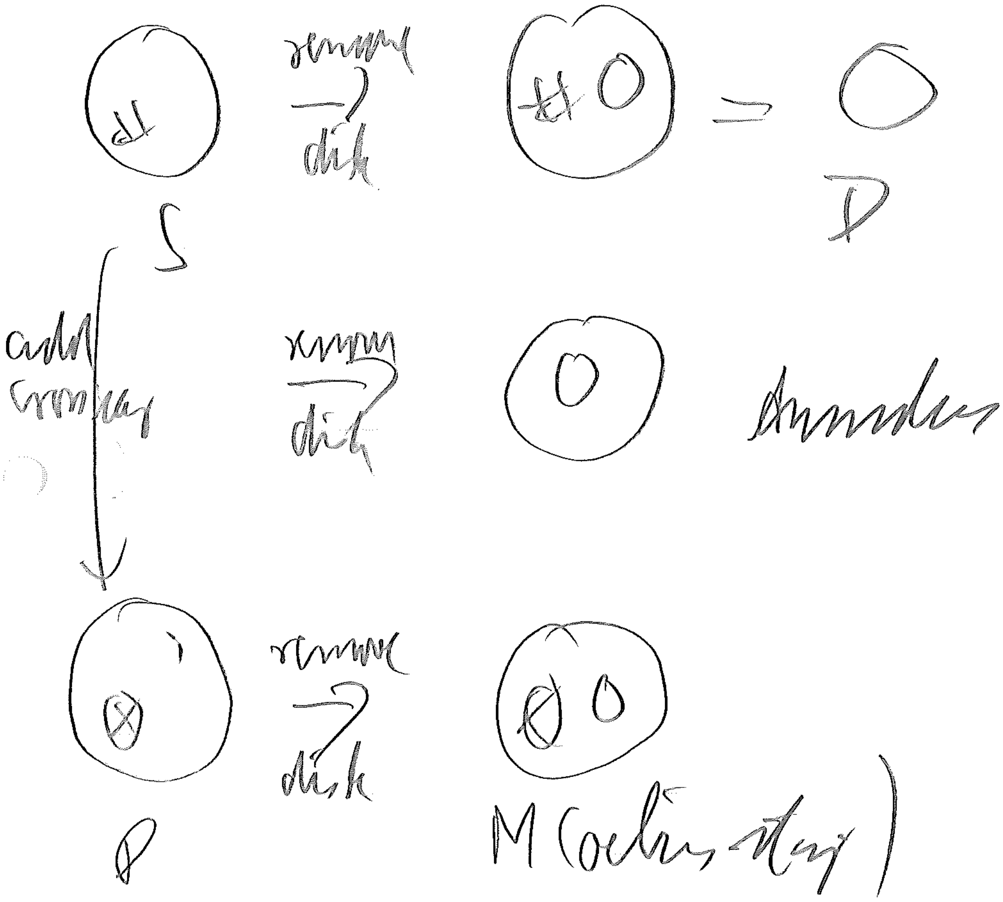
$P \# P = \#^2 P = K$

$P \# P \# P = P \# K = P \# T$

$\left.\right\}$ $\#^k P$, $k \geq 1$ not orientable



Compact SF w/ boundary:
remove disks



Euler Characteristic

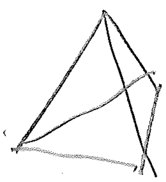
$\chi = |V| - |S| + |T| - \dots$

simplicial
(triangul)

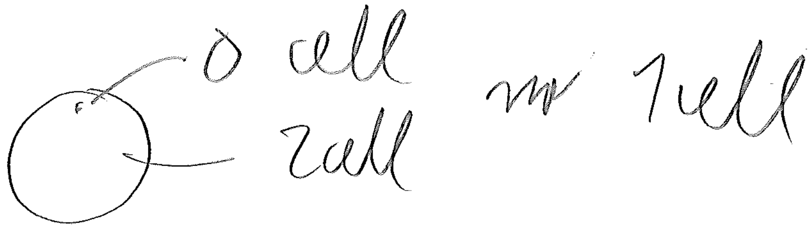
$= |0| - |1| + |2| - \dots$

cell decomposition

S

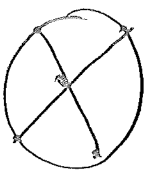


$\simeq$

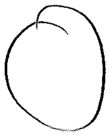


$\chi = 4 - 6 + 4 = 2 = 1 - 0 + 2$

D



$\chi = 5 - 8 + 4 = 1$

$\simeq$  $= 0 - 1 + 1 = 1$

Euler Char (2)

$\chi = b_0 - b_1 + b_2 \cdots$

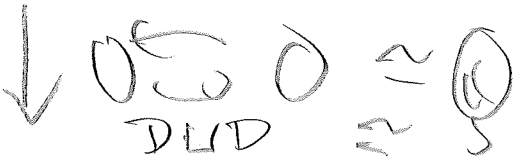
b_n = rank n-th singular homology grp

Disjoint union:

$\chi(M \sqcup N) = \chi(M) + \chi(N)$

Inclusion-exclusion (fine point)

$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N)$



Connected Sum

$\chi(M \# N) = \chi(M) + \chi(N) - \chi(S^n)$

$= 2$ for $n = 2$

$\chi(M \times N) = \chi(M)\,\chi(N)$

$\chi(\tilde{M}) = \deg(\tilde{M} \to M) \cdot \chi(M)$ (unramified cover)

+ ramification $\to$ Riemann-Hurwitz

Euler Char (3)

$\chi(E \xrightarrow{F} B) = \chi(F) \cdot \chi(B)$ (fibre-bundle) fibration

$$\boxed{\begin{aligned} \chi(g,b,c) &= 2-2g-b-c \\ g &= \#\text{ handles } T \\ b &= \#\text{ boundaries (cutting discs)} \\ c &= \#\text{ crosscaps} \end{aligned}}$$

Gauß-Bonnet Thm

extrinsic

$$\chi = \frac{1}{2\pi}\left(\int_M K\,dA + \oint_{\partial M} k\,ds\right)$$

K = Gaussian curvature of M

k = geodesic curvature of ∂M
(deviation from geodesics

$K = \chi_1\chi_2$ $\quad \chi_i$ principal curvature

$$\int_M K\,dA = \int_M R\sqrt{g}\,d^2x$$

Examples

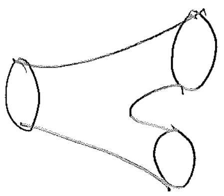
	g	b	c	$\chi = 2-2g-b-c$
S	0	0	0	2
D	0	1	0	1
P	0	0	1	1
T	1	0	0	0
K	0	0	2	0
CA	0	2	0	0
M	0	1	1	0
				⋮

closed SF $\Leftrightarrow$ $\begin{cases} \chi \\ \text{orientable: } y/n \end{cases}$

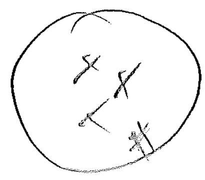
$\downarrow$ cut out discs

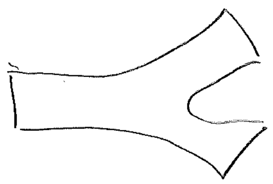
SF w/bd

String couplings

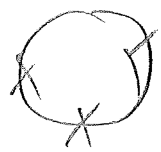


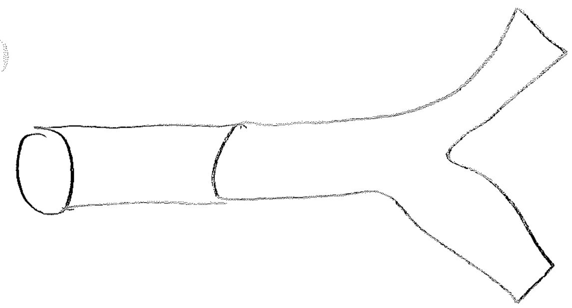
χ_c



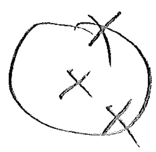


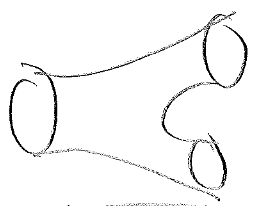
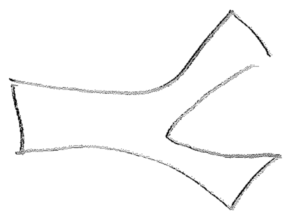
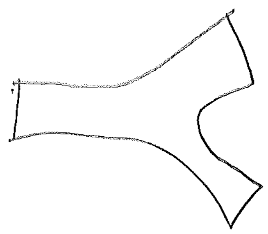
χ_o



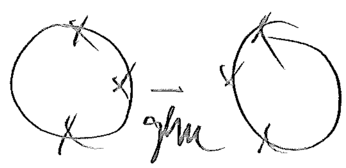


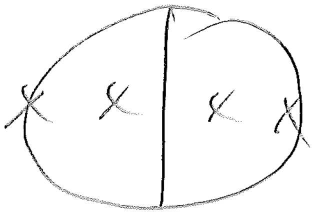
χ_{oc}



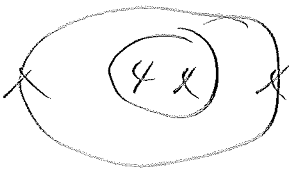
 $\simeq$  $+$ 

$$\boxed{\chi \quad \simeq \quad \chi_o^2}$$

 $\simeq$ 



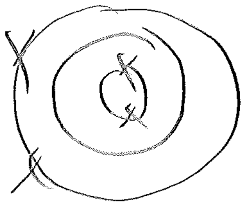
$x_{0c} \cdot x_{0c}$



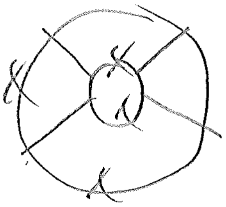
$x_{0c} \cdot x_c$

$\boxed{x_{0c} \simeq x_c}$

check consistency



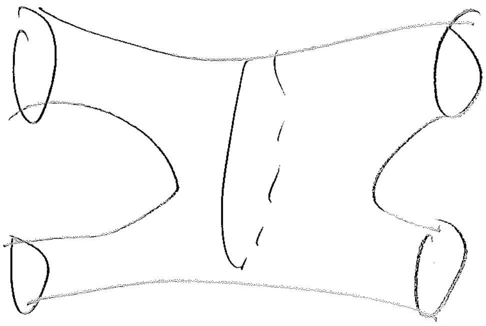
$x_{0c}\, x_{0c}$



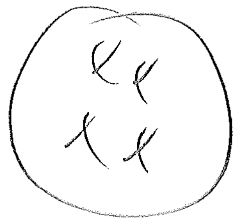
$x_0 x_0 x_0 x_0$

$\boxed{x_{0c} \simeq x_0^2}$

$\Rightarrow \quad x \simeq x_c \simeq x_{0c} \simeq x_0^2$



$\simeq$



$x^2 = \frac{1}{x^2} \cdot x^4$

$4 = |$ext. w/ str. $M|$

$2 = \chi(S)$

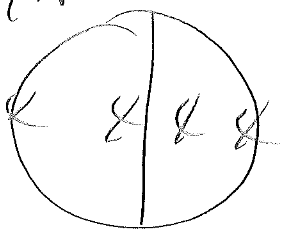
general

$$x^{-\chi(g,b,c)} \, x^M \, x_\partial^N$$

$N = |$open strs, ext strs$|$

Some checks

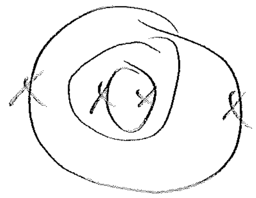
$\Sigma = D_1 \quad M = 2, N = 2$

 $=$  $\quad x_{oc}^2 = x^2$

$$x^{-1} x^2 x_0^2 = \frac{x^3}{x} = x^2$$

$\chi(D) = 1$

$\Sigma = A \qquad M = 0 \quad N = 4$

 $\quad x_{oc}^2 = x^2$

$$x^{-0} x_0^4 = x^2$$

$\chi(A) = 0$
