Selected Topics in String Theory
Liverpool PGR lectures (2022)

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1 Lecture 1: Surface topology and string perturbation theory (31/10/2022)

We reviewed the topological classification of compact surfaces. Closed surfaces are connected sums of tori and real projective planes. Compact surfaces with boundaries result from cutting out discs. Surfaces are classified by the numbers g of handles, b of boundaries and c of cross caps (glued-in Moebius strips). Surfaces are orientable in the absence of cross caps, and non-orientable otherwise.

String amplitudes are given by a sum over all surfaces connecting a given set of line segments (external open strings) and circles (external closed strings). This corresponds to a path integral, with boundary conditions encoding the external states, with measure provided by the Polyakov action. Such a description is potentially off-shell. Standard string perturbation theory defines on-shell scattering amplitudes in terms of compact surfaces (possibly with boundaries) with marked points in the interior or at the boundaries to represent on-shell external states (at the marked points one inserts the vertex operators corresponding to the external on-shell states).

There are three types of string vertices involving: three closed strings, with coupling constant κ_c, three open strings, with coupling κ_o, and two open and one closed strings, with coupling κ_{oc}. Inspection of the corresponding surfaces indicates that (up to numerical factor which can be fixed by actually computing amplitudes)

$$\kappa_c \simeq \kappa_{oc} \simeq \kappa_o^2 \,.$$

The coupling carried by the contribution of a worldsheet surface $\Sigma_{g,b,c}$ with M external closed and N external open strings is:

$$A(1,\ldots,M,1,\ldots,N,g,b,c) \sim \kappa_c^{-\chi(g,b,c)} \kappa_c^M \kappa_o^N$$

where

$$\chi(g,b,c) = 2 - 2g - b - c$$

is the Euler characteristic of $\Sigma_{g,b,c}$. We discussed that the Euler characteristic is a topological invariant, which can be computed, for example, by a simplicial or by a cell decomposition of the surface.

Literature: To prepare this lecture I used the 'obvious' pages on Wikipedia and Scholarpedia, together with my Bad Honnef lecture notes, [1], Section 3.6.

Further reading: Path integrals with boundary conditions (that is, over 'bordered Riemann surfaces') can express off shell amplitudes, see for example [2] and references therein. See also [3] for a treatment by mathematicians.

2 Lecture 2: G structures on surfaces (7/11/2022)

Motivation: String interactions are defined by compact surfaces connecting a set of marked points. Vertex operators located at the marked points represent external states. When computing transition amplitudes (in the Euclidean formulation) surfaces are weighted by the Polyakov action:

$$A(1,\ldots,N) = \int DXdhe^{-S_P[X,h]}V_1\ldots V_N$$

where the path integral DX is over all maps form worldsheets Σ into spacetime (this includes a sum over all possible topologies), and where the path integral Dh is over all metrics on Σ. V_i are integrated vertex operators

$$V_i = \int_\Sigma \mathcal{O}_i(x)\mathrm{vol}_h(x)$$

where $\mathcal{O}_i(x)$ is the local operator corresponding to the external state 'i', and where $\mathrm{vol}_h(x)$ is the volume element associated with the metric h.

Topological aspects were discussed in Lecture 1. We now turn to metrics, and, related to this, to complex structures. We introduce the language of G-structures, which requires a few pre-requisites.

A *fibre bundle* (depending on choice of terminology, also simply called a *bundle*) B over a manifold M is a space which locally (around each point $x \in M$) is the product of an open subset $U \subset M$ and a fibre F.

$$\pi : B \to M \ \text{ projection}$$

$$\forall x \in M\ \exists U \subset M : \pi^{-1}(U) \cong U \times F$$

where U is an open neighbourhood of x.

In other words, we obtain B from M by attaching to each $x \in M$ a space ('fibre') $F \cong \pi^{-1}(x)$. Examples:

- A vector bundle is a fibre bundle where the fibres are vector spaces.
- A G-principal bundle, where G is a group, is a fibre bundle where the fibre is a G-principal homogeneous space. (This means that G acts on F freely and transitively. We can identify F with G by choosing a point on F.)

A section s of a fibre bundle B over an open subset $U \subset M$ is a right inverse of the projection map:

$$s : U \to B\ , \quad \pi \circ s = \mathrm{Id}_U$$

The image $s(U) \subset B$ can be viewed as a graph. A section associates to each point $x \in U \subset M$ an element of the fibre $\pi^{-1}(x) \simeq F$ over x. If B is a vector bundle, then $\pi^{-1}(x)$ is a vector space, and a section associates to each point x a vector v_x. Sections of vector bundles describe vector fields.

Sometimes one includes in the definition of a fibre bundle the action of a group G on the fibre F. Then what was called a fibre bundle above is just called a bundle. The more general concept of a fibration arises when one replaces the local 'triviality' (product structure) by a weaker property. There are different types of fibrations depending on which such condition one chooses.

With any manifold M one gets fibre bundles which are canonically associated to it, that is, defined by M without additional data.

- The tangent bundle TM. This is a vector bundle with fibre $T_xM \cong \mathbb{R}^m$ where $m = \dim M$. Observe this has a natural action by the group $G = GL(m, \mathbb{R})$.

- The frame bundle FM. This is a principal bundle with fibres given by frames, that is bases F_xM for T_xM. Since m linearly independent vectors form a $GL(m, \mathbb{R})$ matrix, this is a $GL(m, \mathbb{R})$ principal bundle.

Given a G-principal bundle and a representation ρ of G one obtains associated vector bundles by choosing F to be the ρ-representation space. The tangent bundle TM is associated to the frame bundle FM by the fundamental $GL(m, \mathbb{R})$ representation. In general, such bundles require extra data, the choice of a G-representation, and are not canonical. The tangent bundle TM is defined through tangent vectors of curves on M, which makes it a natural or canonical bundle.

A G-structure on M is a principal G-sub-bundle of the frame bundle. A G-structure exists if it is possible to consistently deform the fibres of the frame bundle so that all fibres correspond to a subgroup $G \subset GL(m, \mathbb{R})$. This may or may not be possible, in general there may be an obstruction to the existence of a G-structure. Examples:

- $G = GL^+(m, \mathbb{R}) \subset GL(m, \mathbb{R})$. Existence amounts to the question whether we can cover M with coordinate systems which are all right-handed (or all left-handed), that is, whether M is orientable. In general, M need not be orientable (Klein bottle, real projective plane).

- $G = O(m) \subset GL(m, \mathbb{R})$. This amounts to the question whether a Riemannian metric exists on M. This is always the case, because invertible matrices admit a polar decomposition, $GL(m, \mathbb{R}) \cong O(m) \cdot \mathrm{Sym}(m)$, and the set $\mathrm{Sym}(m)$ of symmeric matrices is contractible. In other words $GL(m, \mathbb{R})$ and $O(m)$ are 'topologcially equivalent.'

- $G = O(p, q) \subset GL(p+q, \mathbb{R})$ Corresponds to existence of a pseudo-Riemannian metric of signature (p, q). This is in general obstructed, i.p. not every manifold admits a Lorentzian structure and qualifies as a spacetime.

- $G = SO(m) \subset GL^+(m, \mathbb{R})$. This corresponds to the existence of oriented Riemannian structure (existence of right-handed orthonormal frames). Always possible, if M is orientable.

- $G = GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$. Corresponds to the existence of complex frames covering M. This defines an almost complex structure.

- $U(n) \subset SO(2n) \subset GL(2n, \mathbb{R})$. Corresponds to an almost Hermitian structure, existence of a Riemannian metric compatible with a complex structure. As we see, this requires that M is orientable.

Specifically for surfaces, $\dim M = 2$:

- $O(2) \subset GL(2, \mathbb{R})$. Positive definite metric, exists always.

- $O(1,1) \subset GL(2, \mathbb{R})$. Lorentz metric. Does not exist in general. For closed orientable surfaces Σ_g only exists on torus $T = \Sigma_1$. That's why the Polyakov path integral approach uses Euclidean worldsheet.

- $GL^+(2, \mathbb{R}) \subset GL(2, \mathbb{R})$. Orientable. Closed orientable surfaces are connected sums of spheres and tori, and labeled by their genus g.

- $SO(2) \subset GL^+(2, \mathbb{R}) \subset GL(2, \mathbb{R})$. Orientable Riemannian metric, can work with right-handed orthonormal frames. Comes for free if orientable.

- $GL(1, \mathbb{C}) \subset GL(2, \mathbb{R})$. Complex structure. By polar decomposition

$$GL(1, \mathbb{C}) \cong \mathbb{C}^* \cong \mathbb{R}^+ \cdot SO(2) \sim SO(2)$$

  where $\sim$ denotes contraction, this exists whenever the surface is orientable.

- $U(1) \cong SO(2) \subset GL^+(2, \mathbb{R})$, almost Hermitian structure. If M orientable, then oriented metric is compatible with complex structure.

A Riemann surface is a connected complex manifold of complex dimension one. Hence, it is orientable. A Riemann surface can be non-compact (complex plane $\mathbb{C}$, open disk $\overset{o}{D}$, upper half plane H), closed (compact without boundary) (sphere S, torus T, and connected sums thereof), or compact with boundary (closed disc D). On a Riemann surface, there is a one-to-one correspondence between complex structures (complex manifolds/biholomorphic maps) and conformal structures (metrics/conformal factors).

A space is called simply connected if all closed curves can be contracted to points. Every connected space admits a unique (up to equivalence) simply connected covering space, called the universal cover:

$$\pi : \tilde{M} \to M \ , \quad |\pi^{-1}(x)| \in \mathbb{Z}^+ \cup \{\infty\} \ , \text{const.}$$

Every point on M is covered $m \geq 1$ times, same m for all points.
Example:

$$\pi \ : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z} \ , \quad x \mapsto e^{2\pi i x}$$

is an ∞-to-one map, which covers S^1. $\mathbb{R}$ is simply connected.

Uniformization theorem. If Σ is a simply connected Riemann surface, then Σ is conformally (=biholomorphically) equivalent to one of the following spaces

- The Riemann sphere S.
- The complex plan $\mathbb{C}$.
- The open unit disk $\overset{o}{D}= \{z \in \mathbb{C}|\ |z| < 1\}$, conformally equivalently, the upper half plane $H = \{\tau \in \mathbb{C}|\Im(\tau) > 0\}$.

These three space admit metrics of constant curvature.

- On the Riemann sphere:

$$ds^2 = \frac{|dz|^2}{(1+|z|^2)^2}\ , \quad R > 0$$

($R =$ Ricci scalar.) This so called chordal metric is isometric to the standard round metric on the two-sphere. The metric has constant positive curvature.

- On the complex plane

$$ds^2 = |dz|^2 = dx^2 + dy^2\ , \quad R = 0$$

This is the standard flat metric.

- On the open disk $\overset{o}{D}$:

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}\ , \quad R < 0$$

Equivalently on the upper half plane H:

$$ds^2 = \frac{|d\tau|^2}{(\Im(\tau))^2}\ , \quad R < 0$$

These are the Poincaré metrics for the two standard models of hyperbolic geometry. They have constant negative curvature.

The integrated curvature on a compact manifold is a topological invariant, for surfaces, the Euler characteristic (we only consider closed surfaces, generalization for surfaces with boundary exists):

$$\chi(\Sigma_g) = \frac{1}{4\pi}\int_{\Sigma_g} R\sqrt{h}|dz|^2 = 2 - 2g$$

This is proportional to the Einstein-Hilbert action. Thus the integrated curvature of Σ_g, is negative for $g > 1$, which implies that the universal cover must be the open disk/upper half plane, which admits a metric of constant negative curvature. Moreover, it can be shown that any metric on a closed oriented two-dimensional Riemannian manifold is conformally equivalent to a metric of constant curvature.

Closed Riemann surfaces Σ_g:

- Elliptic = universal cover is the Riemann sphere. $\Sigma_0 \cong S$.
- Parabolic = universal cover is the complex plane. $\Sigma_1 = T \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z})$. (Mod out by two linearly independent translations.)
- Hyperbolic = universal cover is the open disk/upper half plane. $\Sigma_g \cong H/\text{Fuchsian group}$, for $g > 1$. Fuchsian groups are discrete subgroups of $PSL(2, \mathbb{R})$.

'Moduli space' of Σ_g:

$$\mathcal{M}_g = \{\text{complex structures}\} \cong \{\text{conformal structures}\} \cong \{\text{constant curvature metrics}\}$$

The Riemann-Roch theorem implies:

$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 0\,, & \text{for } g = 0\,, \\ 1\,, & \text{for } g = 1\,, \\ 3g-3\,, & \text{for} g > 1\,. \end{cases}$$

References: From memory, with help from Wikipedia ...

3 Lecture 3: Polyakov Path Integral, Faddeev-Popov ghosts (14/11/2022)

3.1 Bosonic Gaussian integrals

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}\,, \Re(a) > 0.$$

Multidimensional:

$$\int d^n x e^{-\frac{1}{2}x^T A x + bx} = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2}b^T A^{-1} b},$$

where A is symmetric and positive definite.
Infinite dimensional: let

$$S[\phi] = \frac{1}{2}\int d^D x \sqrt{h} (\partial_m \phi \partial^m \phi + m^2 \phi^2) = \frac{1}{2}\int d^D x \sqrt{h} \phi (\Delta_h + m^2) \phi \geq 0\,,$$

where

$$\Delta_h \phi = -\frac{1}{\sqrt{h}} \partial_m \left(\sqrt{h} h^{mn} \partial_n \phi \right)$$

is the Laplace operator on functions (note sign convention). With our sign convention Δ_h is positive semi-definite (non-negative). Gaussian integral

$$\int D\phi e^{-S[\phi]} = \det{}^{-1/2}(\Delta_h + m^2)$$

where we used that $\Delta_h + m^2$ is positive definite for $m \neq 0$. For $m = 0$ the operator Δ_h has zero modes (the constant functions), which need to be treated separately.

3.2 Zeta-function determinants

Zeta function. Series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges absolutely for complex s with $\Re(s) > 1$. The function $\zeta(s)$ admits a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$, otherwise holomorphic. Some values:

$$\zeta(-1) = -\frac{1}{12} \ , \quad \zeta(0) = -\frac{1}{2} \ , \quad \zeta'(0) = -\frac{1}{2} \ln(2\pi) \ .$$

This allows to regularise divergent series by using analytic continuation:

$$\left(\sum_{n=1}^{\infty} n \right)_{\zeta} = \zeta(-1) = -\frac{1}{12} \ , \quad \left(\sum_{n=1}^{\infty} 1 \right)_{\zeta} = \zeta(0) = -\frac{1}{2} \ .$$

This also allows to regularise divergent products:

$$\left(\prod_{n=1}^{\infty} a \right)_{\zeta} = a^{(\sum_{n=1}^{\infty} 1)_\zeta} = a^{-1/2} \ .$$

One application is the regularisation of determinants of differential operators with discrete, non-negative spectrum. Consider first a Hermitian matrix A with positive eigenvalues $\lambda_1, \dots, \lambda_N$. Define the zeta-function of A by

$$\zeta_A(s) = \sum_{n=1}^{N} \lambda_n^{-s} \ , \quad s \in \mathbb{C} \ .$$

Compute

$$e^{-\zeta_A'(0)} = \exp\left(-\frac{d}{ds} \left[\sum_{n=1}^{N} \lambda_n^{-s} \right]_{s=0} \right) = \exp\left(\sum_{n=1}^{N} \frac{d}{ds} \left[\exp\left(s \ln \lambda_n\right) \right]_{s=0} \right)$$
$$\exp\left(\sum_{n=1}^{N} (\ln \lambda_n) \ \lambda_n^{-s} \big|_{s=0} \right) = \prod_{n=1}^{N} \lambda_n = \det(A) \ .$$

Now let A be a non-negative self-adjoint differential operator on a Hilbert space with discrete spectrum $\{\lambda_n\}$. Define the zeta-function of A by

$$\zeta_A(s) = \sum_{n}{}' \ \lambda_n^{-s} \ , \quad s \in \mathbb{C} \ ,$$

where the 'prime' on the sum denotes the omission of zero eigenvalues. For elliptic differential operators of degree d on a Riemannian manifold of dimension m one can show that $\zeta_A(s)$ converges absolutely for $\Re(s) > m/d$. Moreover $\zeta_A(s)$ admits a meromorphic continuation to $\mathbb{C}$ and is regular at $s = 0$ [3]. Therefore it makes sense to define the Zeta-function determinant of such a differential operators as

$$\det{}'_\zeta A = e^{-\zeta'_A(0)} .$$

Additional remarks

In general the determinant of a linear map/differential operator is not a number but a section of a determinant line bundle. For the above it is critical that the relevant differential operators had a finite-dimensional kernel and a discrete spectrum. Laplace, Dirac and Dolbeault operators on compact manifolds are of this type. See [3] for more information.

3.3 Polyakov path integral over embeddings at worldsheet fixed metric

We want to compute

$$Z = N \int DX e^{-S_P[X,h]} ,$$

where

$$S_P[X,h] = \frac{1}{2}\int_\Sigma d^2z\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu = \frac{1}{2}\int_\Sigma d^2z\sqrt{h}X^\mu\Delta_h X_\mu =: \frac{1}{2}(X, \Delta_h X) .$$

N is a normalisation factor we keep at our disposal. The integral is over all embeddings X of the worldsheet Σ into a fixed spacetime[1], with a fixed worldsheet metric h. The action can be interpreted as a bilinear form associated with the h-Laplacian $\Delta_h = -\frac{1}{\sqrt{h}}\partial_\alpha\sqrt{h}h^{\alpha\beta}\partial_\beta$.

We follow [4] Chapter 14.2. Fix an orthonormal set of eigenvectors ψ_n for the discrete eigenvalues λ_n:

$$\Delta_h\psi_n = \lambda_n\psi_n , \quad (\psi_m, \psi_n) = \delta_{m,n} .$$

Expand the maps X^μ in eigenvectors, separating the zero modes:

$$X^\mu = \sum_{n=0}^\infty a_n^\mu\psi_n = X_0^\mu + X'^\mu , \quad a_n^\mu \in \mathbb{R} .$$

On a compact worldsheet, the zero modes of the Laplacian on functions are precisely the constant functions (maximum principle for harmonic functions). Therefore, normalisation implies

$$1 = (\psi_0, \psi_0) = \int_\Sigma d^2z\sqrt{h}\psi_0^2 \Rightarrow \psi_0 = \left(\int_\Sigma d^2z\sqrt{h}\right)^{-1/2} = \mathrm{vol}_h(\Sigma)^{-1/2} ,$$

[1]Actually a suitable Sobolev space of maps from the worldsheet to spacetime, see [3].

and $X_0^\mu = a_0^\mu \psi_0$.

The path integral can be carried out as a Gaussian integral in the eigenvector basis:

$$\begin{aligned} Z &= N \int DX e^{-\frac{1}{2}(X,\Delta_h X)} = N \int \prod_{\mu,n} da_n^\mu e^{-\frac{1}{2}\sum_{n,\mu}\lambda_n (a_n^\mu)^2} \\ &= N \left[\int \prod_\mu da_0^\mu \right] \left[\int \prod_\mu \prod_{n\neq 0} da_n^\mu e^{-\frac{1}{2}\sum_{\mu,n\neq 0}\lambda_n (a_n^\mu)^2} \right] . \end{aligned}$$

The Gaussian integral over the non-zero modes evaluates to a power of the product of the non-zero eigenvalues, which we interpret as the zeta-function determinant of the Laplacian:[2]

$$\int \prod_\mu \prod_{n\neq 0} da_n^\mu e^{-\frac{1}{2}\sum_{\mu,n\neq 0}\lambda_n (a_n^\mu)^2} = \prod_\mu \left(\prod_{n}{}'_{\zeta} \lambda_n^{-1/2} \right) = \prod_n{}'_{\zeta} \lambda_n^{-D/2} = (\det{}'_\zeta \Delta_h)^{-D/2} .$$

The integral over the zero modes is an integral over constant maps and therefore proportional to the volume of the target spacetime M. More precisely, if

$$\int \prod_\mu dX_0^\mu = \mathrm{vol}(M)$$

then, using that $X_0^\mu = a_0^\mu \psi_0$:

$$\int \prod_\mu a_0^\mu = \left(\int \prod_\mu dX_0^\mu \right) \psi_0^{-D} = \mathrm{vol}(M) \mathrm{vol}_h(\Sigma)^{D/2} .$$

Combining zero and non-zero modes:

$$Z = N \mathrm{vol}(M) \left(\frac{\det{}'_\zeta \Delta_h}{\mathrm{vol}_h \Sigma} \right)^{-D/2} .$$

$\mathrm{vol}(M)$ is a numerical factor which we can absorb into our normalisation constant N. For non-compact spacetimes, like Minkowski or Euclidean space, $\mathrm{vol}(M)$ is infinite, but this infinite factor can be cancelled by including a factor $(\mathrm{vol}(M))^{-1}$ in N.

3.4 Fermionic Gaussian integrals

One fermionic variable θ, $\theta^2 = 0$:

$$\int d\theta\, \theta = 1 , \quad \int d\theta\, 1 = 0 .$$

[2] Compared to finite-dimensional Gaussians, we omit constant numerical factors which can be absorbed in the normalisation constant N.

$2n$ fermionic variables, $\theta_i, \bar{\theta}_i$, $i = 1, \ldots, n$:

$$\int d\theta_i \theta_j = \delta_{ij} = \int d\bar{\theta}_i \bar{\theta}_j \ , \quad \int d\theta_i = \int d\bar{\theta}_i = 0 \ .$$

For integrals over all variables, choose the following sign convention:

$$\int D\theta D\bar{\theta} \ \prod_{i=1}^{n} (\theta_i \bar{\theta}_i) = \int D\theta D\bar{\theta} \ \theta_1 \bar{\theta}_1 \cdots \theta_n \bar{\theta}_n = 1$$

where

$$D\theta D\bar{\theta} = \pm \prod_{i=1}^{n} d\theta_i \prod_{i=1}^{n} d\bar{\theta}_i$$

Compute:

$$Z = \int D\theta D\bar{\theta} \ e^{-\bar{\theta}^T A \theta} = \int D\theta D\bar{\theta} \ \frac{1}{n!} \left(- \sum_{i,j=1}^{n} \bar{\theta}_i A_{ij} \theta_j \right)^n$$

Among the $(n^2)^n$ terms, only $(n!)^2$ are non-zero, namely those proportional to $\theta_1 \bar{\theta}_1 \cdots \theta_n \bar{\theta}_n$. Using Leibniz formula for an $n \times n$ determinant,

$$\det A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)}$$

where S_n is the group of permutations of n objects, the $(n!)^2$ terms can be collected into $n!$ terms proportional to $\det A$:

$$Z = \int D\theta D\bar{\theta} \ \frac{1}{n!} \ \det A \ n! \ \theta_1 \bar{\theta}_1 \cdots \theta_n \bar{\theta}_n = \det A$$

Fermionic Gaussian functional integral for massless Dirac fermions on $\mathbb{R}^D$, with action

$$S[\psi] = \int d^D \bar{\psi} i \gamma^\mu \partial_\mu \psi \ ,$$

is

$$\int D\psi D\bar{\psi} e^{-S[\psi]} = \det(i\gamma^\mu \partial_\mu) = \det{}^{1/2} \Delta \ .$$

Using: $i\gamma^\mu \partial_\mu i \gamma^\nu \partial_\nu = -\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \partial_\mu \partial_\nu = -\delta^{\mu\nu} \partial_\mu \partial_\nu = \Delta$ (conventional sign). The Hermitian Dirac operator $i\gamma^\mu \partial_\mu$ is the square root of the (non-negative) Laplacian.

3.5 Path integral over metrics

We now attempt to evaluate

$$Z = N \int Dh DX e^{-S_P[X,h]}$$

where we integrate over all metrics on a worldsheet of fixed topology. N is a (possibly infinite) normalisation constant.

Since the action is reparametrisation and Weyl invariant, we may try to impose a gauge

$$h_{\alpha\beta} = e^{2\phi}\hat{h}_{\alpha\beta}$$

so that the integral over metrics factorises and can be absorbed by an infinite normalisation factor. Locally we can choose $\hat{h}_{\alpha\beta}$ to be the standard flat metric $\delta_{\alpha\beta}$.

This requires to parametrise deformations of the metric in terms of vector fields which generate reparametrisations, and of Weyl transformations. We can package deformations of the metric into deformations of the traceless part and of the trace part:

$$\begin{aligned}\delta h_{\alpha\beta} &= -(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha) + 2\Lambda h_{\alpha\beta} \\ &= -(Pv)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta}\end{aligned}$$

Here $2\tilde{\Lambda} = 2\Lambda - \nabla_\gamma v^\gamma$ and

$$P : v_\alpha \mapsto (Pv)_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - (\nabla_\gamma v^\gamma) h_{\alpha\beta}$$

is a differential operator mapping vector fields to rank two, symmetric, traceless tensor fields. The operator

$$P^\dagger : t_{\alpha\beta} \mapsto (P^\dagger t)_\alpha = -2\nabla^\beta t_{\alpha\beta}$$

which maps rank two, symmetric, traceless tensor fields to vector fields is the adjoint of P in the sense that with respect to the scalar products on the respective spaces:

$$(v, P^\dagger t) = (Pv, t) \ .$$

Let us at first ignore complications that arise when $P, P^\dagger$ have zero modes. This amounts to assuming that there is a one-to-one correspondence between vector fields (reparametrisation) and rank 2 tracesless symmetric tensor fields. The integration measure over metrics decomposes into integrations over the traceless and the trace part, which we can rewrite as integrations over reparametrisations (vector fields) and Weyl transformations. This rewriting introduces a Jacobian

$$Dh = D(Pv)D\tilde{\Lambda} = DvD\Lambda|\det P| = DvD\Lambda(\det PP^\dagger)^{1/2} \ .$$

The relation $|\det P| = (\det PP^\dagger)^{1/2}$ generalises a standard relation for complex matrices:

$$\begin{aligned}&\det M^T = \det M \ , \ \ \det M^* = (\det M)^* \Rightarrow \det M^\dagger = (\det M)^* \\ &\Rightarrow \det MM^\dagger = \det M \det M^\dagger = |\det M|^2 \ .\end{aligned}$$

The expression $(\det PP^\dagger)^{1/2}$ manifests the analogy with the above example of a massless Dirac fermion. $P, P^\dagger$ are 'Dirac-like operators', whereas $PP^\dagger$

and $P^\dagger P$ are 'Laplace-like.'[3] More details about the operators $P, P^\dagger$ and their determinants can be found in the references listed at the end of this section.

If we further assume that the integration measure is both reparametrisation and Weyl invariant, then the integration over reparametrisations and Weyl transformations factorises and we obtain

$$Z = N \left(\int Dv \right) \left(\int D\Lambda \right) \int DX |\det P| e^{-S[X, e^{2\phi}\hat{h}]} .$$

The integrals $(\int Dv)$, $(\int D\Lambda)$ are interpreted as the (infinite) volumes of symmetry groups which can be cancelled against the (infinite) normalisation constant N.

The Jacobian determinant can be expressed as a fermionic Gaussian functional integral over an anti-commuting vector field c^γ (ghost) and an anti-commuting rank 2 symmetric traceless tensor field $b_{\alpha\beta}$ (anti-ghost):

$$|\det P| = \int DbDc \, e^{-\frac{1}{2\pi}\int d^2z\sqrt{\hat{h}}\hat{h}^{\alpha\beta} b_{\beta\gamma}\nabla_\alpha c^\gamma} = \int DbDc e^{-S_{\text{ghost}}[b,c,\hat{h}]}$$

Assuming our assumptions to be correct, the path integral would evaluate to

$$Z = \int DbDcDX \, e^{-S_P[X,\hat{h}] - S_{\text{ghost}}[b,c,\hat{h}]} .$$

Here we used the normalisation constant N to cancel the factorized volume of the symmetry group, and we have converted the Jacobian into an action for auxiliary fields, the FP ghosts b, c. Such ghosts fields arise in the quantisation of gauge theories. Their inclusion insures that the gauged fixed quantum theory is still consistent with the underlying local gauge symmetry. Ghost fields have the wrong spin-statistics relation. For bosonic symmetries they are anti-commuting tensor fields.

Complications arise since two of our assumptions are not correct.

- The integration measure is reparametrization invariant but not Weyl invariant.
- The relation between vector fields and rank two symmetric traceless tensor fields is not one-to-one, because the operators P and $P^\dagger$ can have a non-trivial kernel.

3.6 The Weyl anomaly

Since the integration measure is not Weyl invariant, the integration over Weyl transformations does not factorise in general. One obtains

$$Z = \int D\phi DX \, e^{-S_P - S_{\text{ghost}} - (c + c_{\text{ghost}}) S_L[\phi]}$$

where $S_L[\phi]$ is the so-called Liouville action. One has two options:

[3]They are examples of first and second order elliptic differential operators, respectively, see [4], [3].

1. Cancel the ‘conformal anomaly’ by choosing the central charge c of the CFT defined by the gauge fixed Polyakov action to be minus the central charge of the ghost CFT. Since $c_{\text{ghost}} = -26$, this amounts to fixing the spacetime dimension to the critical value $D = c = 26$. This is the critical string theory.

2. We can accept that in the quantum theory there is a new degree of freedom whose dynamic is governed by the Liouville action. The combined worldsheet theory of embeddings, ghosts and Liouville mode is still a conformal field theory. The dynamics of the Liouville mode is complicated, in particular it does not allow standard vacua with constant ϕ and thus does not provide an easy way to construct string theories with lower-dimensional Minkowski vacua. See [5] Chapter 9.9 for more information about non-critical string theory.

We restrict ourselves to critical string theories.

3.7 Zero modes

Zero modes arise in two ways:

1. Solutions of $(Pv)_{\alpha\beta} = 0$ are conformal Killing vector fields, that is vector fields which do not change the conformal structure. If we integrate over all vector fields, including the conformal Killing vector fields, we over-count metrics by a factor which is equal to the (possibly infinite) volume of the global conformal group. This can be handled by additional gauge fixing and cancelling the group volume against a normalisation factor.

2. Solutions of $(P^\dagger t)_\alpha = 0$ are rank two symmetric traceless tensor fields which cannot be mapped to vector field. In other words, they correspond to deformations of the metric which cannot be generated by vector fields. This situation arises when no global gauge fixing of the form

$$h_{\alpha\beta} = e^{2\phi}\hat{h}_{\alpha\beta}$$

is possible. On a closed Riemann surface Σ_g every metric is globally conformal to a metric of constant curvature, and for $g > 0$ there is a finite-dimensional family of such metrics. To perform an integration over all metrics we need to integrate over this family. This is equivalent to integrating over all conformal structures on Σ_g, which in turn is equivalent to integrating over all complex structures on Σ_g. In the path integral one integrates over the moduli space $\mathcal{M}_g$ of complex structures. $\mathcal{M}_g$ is itself a complex space equipped with a natural metric and (Weil-Peterson) measure $d\mu(m_i)$, where the ‘moduli’ m_i are certain complex coordinates on $\mathcal{M}_g$.

As a result, the path integral takes the form

$$Z = N\mathrm{Vol}_{\text{Global Conformal}} \int DbDcDX d\mu(m) e^{-S_P[X,\hat{h}(m_i)] - S_{\text{ghost}}[b,c,\hat{h}(m_i)]}$$

What remains to study?

- Study the CFT defined by the combined Polyakov and ghost action on the surfaces $(\Sigma_g, \hat{h}(m_i))$. This is a non-unitary CFT with central charge 0. The ghost fields keep track of quantum gauge invariance.
- For string theory purposes, integrate over the moduli space $\mathcal{M}_g$ and sum over all topologies, parametrised by g (for closed oriented strings).

Literature:
Good first read: [6] Chapter 3.4, 6. Includes worldsheets with boundaries and non-orientable worldsheets.
With more mathematical details: [4] Chapter 14.
By mathematicians: [3]. Mathematically rigorous treatment of infinite-dimensional integrals and of determinants of differential operators, includes detailed discussion of the Liouville mode.

4 Lecture 4: First oder systems in CFT, aka bc-systems (21/11/2022)

In the last lecture we saw that the CFT of the critical bosonic string is given by the string coordinates X^μ together with the FP ghosts b, c, which carry central charge $c_{b,c} = -26$. To extend our discussion to superstrings we now introduce worldsheet fermions ψ^μ, each of which carries central charge $c = 1/2$. Starting with a theory with local supersymmetry on the worldsheet, gauge fixing leads to additional FP ghosts β, γ which are worldsheet spinor fields with Bose statistics. Since their central charge is $c_{\beta,\gamma} = 11$, the combined central charge of ghosts b, c and superghosts β, γ is $c_{\text{ghost}} = -15$. For critical superstrings this cancels against the central charge of the dynamical degrees of freedom X^μ, ψ^μ which fixes the critical dimension $D = 10$.

We started with 'real' chiral spinors $\psi(z)$. This properly means: fields which have been obtained by the Euclidean continuation of Majorana-Weyl spinors. These fields have weight $h = 1/2$ and generate a CFT with $c = 1/2$. Their equation of motion is first order: $\bar{\partial}\psi = 0$.

Given two real fermions we can combine them into complex conjugate fermions $\psi^\pm$, which together form a CFT with $c = 1$. This is referred to as the CFT of a complex fermion.

Can we really have half-integer conformal weights $h = 1/2$? To define such fields consistently on a worldsheet Σ_g, this requires the existence of a line bundle, called a spinor bundle S, which is the square root of the cotangent bundle in the sense that $S \otimes S \cong T^*\Sigma_g$.

Given an open cover $O = \{O_a\}$ of Σ_g, transition functions t_{ab} must satisfy the consistency condition

$$t_{ab}t_{bc}t_{ca} = 1$$

on triple overlaps $O_a \cap O_b \cap O_c$. For the cotangent bundle this follows from the existence of an atlas of the underlying manifold. A spin bundle S has transition functions s_b which are square roots of the transition functions t_{ab}: the transition functions $\frac{dw}{dz}$ relating co-vectors, are just the derivatives of the coordinate transformations $z \mapsto w$ on Σ_g.

To define fields with weight $h = 1/2$, we need to be able to consistently take square roots of $(\frac{dw}{dz})^{1/2}$. Since taking square roots involves the choice of a sign, the existence of transition functions on $T^*\Sigma_g$ only guarantees that

$$\tilde{s}_{ab}\tilde{s}_{bc}\tilde{s}_{ca} = w_{abc} \in \{1, -1\} = \mathbb{Z}_2$$

where $\{w_{abc}\}$ is a family of $\mathbb{Z}_2$ valued constant functions on the triple overlaps. The existence of consistent transition functions s_{ab}

$$s_{ab}s_{bc}s_{ca} = 1$$

requires the existence of a family $\{c_{ab}\}$ of constant $\mathbb{Z}_2$-valued functions on double overlaps $O_a \cap O_b$, which satisfy

$$c_{ab}c_{bc}c_{ca} = w_{abc}$$

This means that the Cech 2-co-chain $\{w_{abc}\}$ must be a co-boundary. Cech n chains are constant functions on overlaps of $n + 1$ elements of our covering. One can define a nilpotent co-boundary operator, which maps n-chains to $n + 1$ chains. n co-chains which are in the kernel of the co-boundary operator are called n co-cycles, and those which are images of $(n - 1)$-co-chains are called co-boundaries. The quotient of n-cocycles modulo n coboundaries is called the n cohomology group.

This is the same structure (a cochain complex) as we have for differential n-forms and the exterior derivative d, leading to the de Rham cohomology groups $H_n(\Sigma_g, \mathbb{R})$. In our case the co-chains are not differential forms, but functions on overlaps, and linear combinations are taken with coefficients in $\mathbb{Z}_2$ rather than $\mathbb{R}$.

$\{w_{abc}\}$ is a Cech 2-co-cycle and thus defines a element

$$[\{w_{abc}\}] \in H^2(O, \mathbb{Z}_2)$$

in the second Cech cohomology group (with values in $\mathbb{Z}_2$ of the cover O. If O is a good cover, then $[\{w_{abc}\}]$ does not depend on the cover and defines an element

$$[\{w_{abc}\}] \in H^2(\Sigma_g, \mathbb{Z}_2)$$

of the second $\mathbb{Z}_2$-valued cohomology group of Σ_g, called the second Stiefel-Whitney class of Σ_g. To guarantee that $[\{w_{abc}\}]$ is a co-boundary we require that the second Stiefel-Whitney class is trivial. For surfaces the second Stiefel-Whitney class is equal modulo 2 to the Euler class, and therefore vanishes for all closed Riemann surfaces Σ_g. This implies the existence of spinor bundles and spinor fields.

Regarding uniqueness. Inequivalent choices of $\{c_{ab}\}$ are labeled by elements of $H^1(\Sigma_g, \mathbb{Z}_2) \cong H_1(\Sigma_g, \mathbb{Z}_2)$. The related group $H_1(\Sigma_g, \mathbb{Z})$ is the first singular homology group of Σ_g. Singular homology groups are based on continuous maps from simplizes into the manifold Σ_g. (This version of homology is a apparently more poweful then simplicial homology, which is based on triangulations). Chains are formal $\mathbb{Z}$ linear combinations of such maps, and one can define a nilpotent boundary operator. More intuitively, chains are formal integer linear combinations of points (0-chains), curves (1-chains) and surfaces (2-chains), and the first homology group is the group of equivalence classes of closed curves modulo curves which are boundaries of surfaces. For closed Riemann surfaces this group is $H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and its elements can be interpreted as winding numbers along the generating curves. If one replaces $\mathbb{Z}$ by $\mathbb{Z}_2$ then $H_1(\Sigma_g, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ and the numbers ± 1 associated with the generating curves encode periodic or antiperiodic boundary conditions for spinor fields. On Σ_g there are 2^{2g} independent choices of boundary conditions, which define 2^{2g} equivalence classes of spin bundles ('spin structures'). Thus when dealing with spinor fields we have to specify boundary conditions. When working in the complex plane we either use periodic (NS) or anti-periodic (R) boundary conditions along curves which enclrcle the origin $z = 0$ once.

To include FP ghost fields, we consider more general systems with first order equations of motion. In the following b, c are generic fields with first order equations (bc-system). The same symbols are used for the FP-ghosts of the bosonic string which are a special case thereof. b, c are generalizations of $\psi^{\pm}$ in the following sense.

1. We allow any weights $\lambda, \lambda' \in \frac{1}{2}\mathbb{Z}$ which are consistent with scale invariance. This requires that if b has weight λ, then c has weight $\lambda' = 1 - \lambda$.
2. We allow any combination between tensors/spinors (integer/half-integer weight) and Bose/Fermi statistics (commuting/anti-commuting fields). We denote Fermi statistics by $\epsilon = 1$ and Bose statistics by $\epsilon = -1$. Fields with the 'wrong' spin-statistics relation are admitted to describe FP ghosts.

For $\lambda \in \mathbb{Z} + \frac{1}{2}$, we admit periodic and anti-periodic boundary conditions.

A bc-system has a $U(1)$ symmetry, which is in general anomalous. b, c carry 'ghost' charges $-1, +1$. The OPE between the energy momentum tensor $T(z)$ and the ghost current $j(w)$ contains an anomalous higher order term $Q/(z-w)^3$ where

$$Q = \epsilon(1 - 2\lambda)$$

is called the background charge. (Explanation below). As a consequence, conservation of the ghost current is violated:

$$\nabla^z j_z \propto Q R[h]$$

where $R[h]$ is the Ricci scalar of the worldsheet metric h. The background is zero iff $\lambda = 1/2$ which is the case of standard fermions $\psi^{\pm}$. The central charge of the bc system is

$$c = \epsilon(1 - 3Q^2)$$

It takes the value $c = 1$ for anti-commuting spinors $\psi^{\pm}$: $\epsilon = 1, \lambda = 1/2$.

In the algebra of modes, the anomaly shows as

$$[L_m, j_n] = -m j_{m+n} + \frac{1}{2} Q n(n+1)\delta_{m+n,0}$$

and affects j_n for $n \neq 0, -1$. While $j_n^\dagger = -j_{-n}$, the operator j_0 has a normal ordering ambiguity. $j_0^\dagger$ is determined by consistency:

$$j_0^\dagger = (-[L_{-1}, j_1])^\dagger = \cdots = -j_0 - Q$$

Denote by $|q\rangle$ a state of ghost charge q, that is $j_0|q\rangle = q|q\rangle$. Then

$$\langle q'|q\rangle \neq 0 \Leftrightarrow q' + q = -Q \ .$$

This explains why Q is called a 'background charge.' Charge on a closed surface must add up to zero by Gauss law. For $Q \neq 0$ the charges associated to the two states whose overlap we are calculating don't add up to zero, indicating a the presence of a further charge Q in the system. Put differently, charge conservation is violated by an amount Q between in and out states.

To show the above relation, it is helpful to consider the more general case of an operator O of ghost charge p, that is, $[j_0, O] = pO$ and to consider its matrix elements $\langle q'|O|q\rangle$. One can show that if $\langle q'|O|q\rangle$, the charges must be related by $p = -(q + q' + Q)$. Above we considered the special case where $p = 0$.

Literature: For fermions and bc systems I have been following [6] closely, though the material of this lecture is distributed over different chapters there. For spin structures and cohomology, see [4].

5 Lecture 5: Bosonization, Lie algebras, and Lattices

Complex fermion $\psi^{\pm}(z)$:

$$\psi^+(z)\psi^-(w) = \frac{1}{z-w} + :\psi^+(w)\psi^-(w): + O(z-w)$$

$\phi(z)$ boson, then

$$e^{i\phi(z)} e^{-i\phi(w)} = \frac{1}{z-w} + i\partial\phi(w) + O(z-w)$$

NS boundary conditions for $\psi^{\pm}(z)$:

$$\psi^{\pm}(ze^{2\pi}) = +\psi^{\pm}(z) \Leftrightarrow \phi(ze^{2\pi i}) = \phi(z) + 2\pi$$

Boson $\phi(z)$ takes values in a circle of radius $R = 1$, $S^1_{R=1} = \mathbb{R}/2\pi\mathbb{Z}$.

For R boundary conditions:

$$\psi^{\pm}(ze^{2\pi}) = -\psi^{\pm}(z) \Leftrightarrow \psi^{\pm}(z) = e^{\frac{i}{2}\phi(z)} \ , \ \ \phi(ze^{2\pi i}) = \phi(z) + 2\pi$$

This leads to an equivalence of CFTs. Partition functions:

$$Z^{\text{Boson}}_{S^1_R}(\tau,\bar{\tau}) = Z^{\text{Dirac fermion}}(\tau,\bar{\tau})$$

Boson valued on unit circle is equivalent to complex fermion (Dirac fermion). We have combined left and right movers. Bosonic partition functions includes sum over windings, fermionic partition function includes sum over spin structures. Same $(c_L, c_R) = (1,1)$ CFT can be written in terms of different fields $\psi^{\pm}$ or ϕ.

Lit: [7]

$2n$ real fermions (n complex fermions) $\cong$ n bosons:

$$\psi^{\pm l} = e^{\pm i\phi^l} ?$$

Issue:

$$:\psi^{\pm l}(z)\psi^{\pm k}(z) := -:\psi^{\pm k}(z)\psi^{\pm l}: , \quad k \neq l$$

while

$$:e^{\pm i\phi^k(z)}e^{\pm i\phi^l(z)} :=: e^{\pm i\phi^l(z)}e^{\pm i\phi^k(z)} :$$

Therefore

$$\psi^{\pm k}(z) = e^{\pm i\phi^k(z)} c_{\pm k}$$

where

$$c_{\pm k}c_{\pm l} = -c_{\pm l}c_{\pm k} \ , \ \ k \neq l$$

The $(\pm)$-valued functions $c_{\pm k}$ are Klein factors/Cocycle factors which convert the mixed commutation/anticommutation relations satisfied by the modes of bosonic exponentials into proper anticommutation relations.

Background: in four dimensions the Spin-Statistics Theorem (Pauli-Lüders Theorem) states that if we assume that a QFT satisfies causality (no action faster than speed of light), then fields with integer spin must be quantized using commutation relations, leading to Bose-Einstein statistics, while fields with half-integer spin must be quantized using anticommutation relations, leading to Fermi-Dirac statistics. More precisely, the relations between fields can be brought to this standard form by a so-called Klein transformation.

Lit: [8].

In $D < 4$ more general types of statistics are possible, and in $D = 2$ fermions and bosons can be equivalent to each other. Bringing OPE/(anti-)commutation relations to standard form may involve a Klein transformation. Finding the Klein transformation can be viewed as a group theoretical problem, hence Klein factors are also referred to as cocycle factors.

Monomials $:\psi^{\pm i_1}(z)\cdots\psi^{\pm i_n}(z):$ in fermions, and their derivatives , are bosonized by expressions of the form

$$e^{i\vec{\lambda}\cdot\vec{\phi}} c_{\vec{\lambda}}$$

where the entries in the vectors $\vec{\lambda}$ are integers for NS boundary conditions and half-integers for R boundary conditions. The vectors λ therefore span the weight lattice of the Lie algebra $D_n \cong \mathfrak{so}(2n)$, which is the Euclidean continuation of the Lorentz Lie algebra. Here $c_{\vec{\lambda}}$ are cocycle factors which ensure that the operators satisfy the correct type of relation. Operators containing an even number of fermions are bosonic operators and must satisfy commutation relations. One example of bosonic operators are the fermion bilinears $:\psi^{\pm l}(z)\psi^{\pm k}(z):$, which are the conserved currents associated with the Euclidean continuation $\mathfrak{so}(2n)$ of the Lorentz Lie algebra.

This is an example of a more general construction, the Frenkel-Kac construction. Let $\mathfrak{g}$ be a simple Lie algebra. Then $\mathfrak{g}$ admits a basis

$$H^i, i = 1, \ldots, l = \mathrm{rank}(\mathfrak{g}) \quad E^\alpha \ , \alpha \in \{\text{Roots of } \mathfrak{g}\} = \Delta$$

where

$$\dim \mathfrak{g} = l + |\Delta|$$

such that

$$\begin{aligned} [H^i, H^j] &= 0 \\ [H^i E^\alpha] &= (e_i \cdot \alpha) E^\alpha \\ [E^\alpha, E^\beta] &= \begin{cases} \epsilon(\alpha,\beta) E^{\alpha+\beta} \ , & \text{if } \alpha+\beta \in \Delta \\ \alpha \cdot \vec{H} & \text{if } \alpha + \beta = 0 \\ 0 \ , & \text{else .} \end{cases} \end{aligned}$$

Here we use that $\alpha \in \Delta$ can be interpreted as vectors in $\mathbb{R}^l$, with ONB e_i, and we have set $\vec{H} = (H^1, \ldots, H^l)$.

Literature: [9].

A simple Lie algebra is called simply-laced if all roots have the same length, conventionally normalized as $\alpha \cdot \alpha = 2$. In this case scalar products between roots take values $\pm 2, \pm 1, 0$ and $\alpha + \beta \in \Delta$ iff $\alpha \cdot \beta = -1$ and $\alpha + \beta = 0$ iff $\alpha \cdot \beta = -2$. The simple simply laced Lie algebras are $A_l \simeq \mathfrak{su}(l+1)$, $D_l \simeq \mathfrak{so}(2l)$, E_6, E_7, E_8. Therefore simply laced Lie algebras are also called ADE Lie algebras.

Given a system of n chiral bosons, we have conserved currents

$$H^k(z) = i\partial X^k(z)$$

Moreover, if the scalars $X^k(z)$ take values in an l-dimensional torus such that

$$E^\alpha(z) = e^{i\alpha\cdot X(z)} c_\alpha$$

are single valued, and where $\alpha \in \Delta$ are the roots of an ADE Lie algebra $\mathfrak{g}$, then $E^\alpha(z)$ are additional conserved currents. We can project out modes by contour integration:

$$H^i_m = \oint \frac{dz}{2\pi i} z^m H^i(z) \ , \quad E^\alpha_m = \oint \frac{dz}{2\pi i} z^m E^\alpha(z)$$

They can be shown to satisfy the following commutation relations:

$$\begin{aligned}
[H^i_m, H^j_n] &= km\delta^{ij}\delta_{m+n,0} \\
[H^i + mE^\alpha_n] &= (e_i \cdot \alpha)E^\alpha_{m+n} \\
[E^\alpha_m, E^\beta_n] &= \begin{cases} \epsilon(\alpha,\beta)E^{\alpha+\beta}_{m+n}\ , & \text{if } \alpha+\beta \in \Delta \\ \alpha \cdot \vec{H}_{m+n} + km\delta_{m+n,0} & \text{if } \alpha+\beta = 0 \\ 0\ , & \text{else}\ . \end{cases}
\end{aligned}$$

where in the explicit construction $k = 1$, but in general k is a central element of the infinite-dimensional, $\mathbb{Z}$-graded Lie algebra generated by $\{T^a_m, k\} \leftrightarrow \{H^i_m, E^\alpha_m, k\}$. Using the notation T^a_m, $a = 1, \ldots, \dim \mathfrak{g}$, $m \in \mathbb{Z}$ the algebra takes the form

$$[T^a_m, T^b_n] = if^{ab}_c T^c_{m+n} + km\delta^{ab}\delta_{m+n,0}$$

where f^{ab}_c are structure constants of the simple Lie algebra $\mathfrak{g}$ (with respect to Hermitian generators).

If the algebra generated by $\{T^a_m, k\}$ was a symmetry algebra, it would need to commute with the Virasoro algebra. However the commutators

$$[L_m, T^a_n] = -nT^a_{m+n}$$

show that only the 'degree 0 subalgebra' generated by T^a_0, which is isomorphic to $\mathfrak{g}$, is a symmetry algebra. This shows that l bosons valued on a suitably chosen torus have a non-abelian $\mathfrak{g}$ symmetry. In this way non-abelian gauge symmetries can be realized in theories of closed strings.

In general, the Virasoro algebra acts on the T^a_m. The operator $d = -L_0$ acts by

$$[d, T^a_n] = nT^a_n$$

The Lie algebra $\hat{\mathfrak{g}}$ generated by $\{T^a_m, k, d\}$ is called the untwisted affine Kac-Moody algebra associated to the simple Lie algebra $\mathfrak{g}$. The algebra $\tilde{\mathfrak{g}}$ generated by T^a_m is the loop algebra associated to $\mathfrak{g}$, that is the Lie algebra of the loop group which is the Lie group of maps $S^1 \to G$, from the circle to a Lie group G (with Lie algebra $\mathfrak{g}$). By adding k one obtains a central extension of the loop algebra, and by further adding d a semi-direct extension of the centrally extended loop algebra. The resulting Kac-Moody algebra can be characterized by a generalized Cartan which has dimension $l+1$ and rank l. See [10] for the general theory of Kac-Moody algebras.

The Virasoro and Kac-Moody algebra form a semi-direct product $\text{Vir} \ltimes \hat{\mathfrak{g}}$. This is an example of an extended chiral algebra. The Hilbert space of the CFT decomposes into irreducible representations of this algebra, and the CFT is completely determined by the OPEs between chiral primaries of the extended algebra. The Kac-Moody algebra $\hat{\mathfrak{g}}$, which is not a symmetry algebra, is a 'spectrum generating algebra.'

A representation Φ of a simple Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$H^i \mapsto \Phi(H^i)\ , \quad E^\alpha \mapsto \Phi(E^\alpha)$$

from $\mathfrak{g}$ to linear operators on some vector space V. Finite dimensional representations of $\mathfrak{g}$ decompose as

$$V = \bigoplus_{w \in W(\Phi)} V_w$$

where $W(\Phi)$ is the set of weights of the representation Φ. Representatives of the basis H^i, E^α act as follows:

$$\begin{aligned}
\Phi(H^i)v_w &= w^i v_w \ , \ \ v_w \in V_w \\
\Phi(E^\alpha)v_w &= \begin{cases} 0 & \text{if } \alpha + w \notin W(\Phi) \\ \in V_{\alpha+w} & \text{if } \alpha + w \in W(\Phi) \end{cases}
\end{aligned}$$

Observe that the difference between any two weights of a representation must be a root.

The lattice Γ_W generated by the weights of all representations of $\mathfrak{g}$ is called the weight lattice. The lattice Γ_R generated by the roots of $\mathfrak{g}$ is called the root lattice. Since the roots are the weights of adjoint representation, $\Gamma_R \subset \Gamma_W$. For ADE Lie algebras $\Gamma_W = \Gamma_R^*$. The dual Γ^* of a lattice Γ is defined by

$$\Gamma^* = \{v \in \mathbb{R}^l | v \cdot w \in \mathbb{Z} \ \forall w \in \Gamma\}$$

A lattice is called integral, if $\Gamma \subset \Gamma^*$. Hence, Γ_R is an integral lattice. A lattice is called selfdual if $\Gamma = \Gamma^*$. A lattice Γ is called even, if

$$v \cdot v \in 2\mathbb{Z} \ \forall v \in \Gamma$$

ADE root lattices are even integral lattices.

Since $\Gamma_R \subset \Gamma_W$ we can form the quotient Γ_W/Γ_R. This is a finite abelian group with vector addition modulo roots. It is called the group of conjugacy classes of representations, reflecting that the weight lattice decomposes into a finite number of equivalence classes modulo the root lattice.

Further note that for $v, w \in \Gamma_W$ and $\alpha, \beta \in \Gamma_R$:

$$(v + \alpha) \cdot (w + \beta) = v \cdot w + \alpha \cdot w + v \cdot \beta + \alpha \cdot \beta \equiv_1 v \cdot w$$

Here we use that $\Gamma_W = \Gamma_R^*$ and $\Gamma_R \subset \Gamma_R^*$. Thus the scalar product between weights depends modulo $\mathbb{Z}$ only on the class in Γ_W/Γ_R.

Moreover

$$(v + \alpha) \cdot (v + \alpha) = v \cdot v + 2v \cdot \alpha + \alpha \cdot \alpha \equiv_2 v \cdot v$$

where we used that ADE root lattices are even lattices. Thus the square norm of a weight vector depends modulo 2 only on the class in Γ_W/Γ_R.

As an example we consider the D_n weight lattice, which is the lattice relevant for bosonization. It has four conjugacy classes.

1. The weights of the adjoint representation are

$$(\pm 1, \pm 1, 0, \cdots, 0), (\pm 1, 0, \pm 1, 0, \ldots, 0), \ldots$$

that is, all but two entries are 0, and two entries are ± 1. There are $2n(n-1)$ such vectors. Since D_n has rank n, the corresponding ladder operators and Cartan generators add up correctly to the dimension $\frac{1}{2}2n(2n-1)$ of $\mathfrak{so}(2n)$. The lattice generated by the weights of the adjoint representation is the root lattice of D_n:

$$\Gamma_R = D_n^{(0)} = \{\lambda \in \mathbb{R}^n | \lambda_i \in \mathbb{Z}\,, \sum_i \lambda_i \in 2\mathbb{Z}\}$$

2. The weights of the fundamental or vector representation are

$$(\pm 1, 0, \ldots, 0), (0, \pm 1, 0, \ldots, 0), \ldots (0, \ldots, 0, \pm 1)$$

That is one entry is ± 1 and the others are 0. There are $2n$ such vectors, matching the dimension $2n$ of the vector or fundamental representation of $\mathfrak{so}(2n)$. The lattice generated by these weights is

$$D_n^{(v)} = D_n^{(0)} + (1, 0, \ldots, 0) = \{\lambda \in \mathbb{R}^n | \lambda_i \in \mathbb{Z}\,, \sum_i \lambda_i \in 2\mathbb{Z} + 1\}$$

3. The weights of the positive chirality Weyl spinor representation are

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots \pm\frac{1}{2}\right)\;, \quad \text{even number of } -$$

There are 2^{n-1} such vectors, matching the dimension of a Weyl spinor representation of $\mathfrak{so}(2n)$. The lattice generated by these spinor weights is

$$D_n^{(s)} = D_n^{(0)} + \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$$

4. The weights of the negative chirality Weyl spinor representation are

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots \pm\frac{1}{2}\right)\;, \quad \text{odd number of } -$$

There are 2^{n-1} such vectors, matching the dimension of a Weyl spinor representation of $\mathfrak{so}(2n)$. The lattice generated by these spinor weights is

$$D_n^{(c)} = D_n^{(0)} + \left(\frac{1}{2}, \ldots, -\frac{1}{2}\right)$$

The group structure of $\Gamma_W/\Gamma_R = \{(0), (v), (s), (c)\}$ can be worked out by adding weights modulo roots:

$$\langle (0), (v), (s), (c) \rangle = \left(\begin{array}{ll} \mathbb{Z}_4 = \langle (s) \rangle = \langle (c) \rangle & \text{for } n \text{ odd} \\ \mathbb{Z}_2' \times \mathbb{Z}_2'' = \langle (s), (c) \rangle & \text{for } n \text{ even} \end{array} \right.$$

As an application, we list the connected Lie groups with Lie algebra $D_{16} \cong \mathfrak{so}(32)$. For any (real, compact) simple Lie algebra $\mathfrak{g}$ there exists a unique (up to isomorphism) simply connected Lie group G with Lie algebra $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{so}(32)$ this group is called Spin(32) and has centre $Z = \mathbb{Z}_2' \times \mathbb{Z}_2''$ and fundamental group $\pi_1(\mathrm{Spin}(32)) = \{1\}$.

This reflects that the group Γ_W/Γ_R of conjugacy classes is isomorphic to the centre Z of the simply connected Lie group G with Lie algebra $\mathfrak{g}$ [11]. Moreover, all other Lie groups with Lie algebra $\mathfrak{g}$ can be obtained as quotients G/C of G by central subgroups $C \subset Z$. Taking such quotients leads to groups with smaller centre but larger fundamental group: the smaller the centre, the more multiply connected the group. Moreover, while for the simply connected group G Lie algebra representations for all classes in Γ_W/Γ_R exponentiate to Lie group representations, only some conjugacy classes are (proper) representations of the groups G/C, while the remaining classes only give projective representations, that is, representations up to sign. Up to isomorphism, the connected Lie groups with Lie algebra $\mathfrak{so}(32)$ are

1. Spin(32) with $Z = \mathbb{Z}_2' \times \mathbb{Z}_2''$ and $\pi_1 = \{1\}$ (simply connected). Representations: $(0), (v), (s), (c)$.

2. $\mathrm{Spin}(32)/\mathbb{Z}_2' \cong \mathrm{Spin}(32)/\mathbb{Z}_2''$. with centre $\cong \mathbb{Z}_2$ and fundamental group $\cong \mathbb{Z}_2$. Representations $(0), (s)$ or $(0), (c)$. The corresponding lattices $D_{16}^{(0),(s)} \cong D_{16}^{(0),(c)}$ are even and self-dual. Modular invariance requires the lattice defining a ten-dimensional heterotic string theory to be even and self-dual. There are two such lattices, up to isometry, in 16 (Euclidean) dimensions: the root lattice of $E_8 \oplus E_8$ and the lattice $D_{16}^{(0),(s)} \cong D_{16}^{(0),(c)}$ generated by the roots and by the weights of one of the two spinor reps of $SO(32)$. This explains why the gauge groups of ten-dimensional supersymmetric heterotic string theories are $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$. We note in passing that the root lattice of E_8 is, up to isomorphism, the only eight-dimensional even self-dual lattice.

3. $SO(32) = \mathrm{Spin}(32)/\mathbb{Z}_2^{(\mathrm{diag})}$, where $\mathbb{Z}_2^{(\mathrm{diag})} \subset \mathbb{Z}_2' \times \mathbb{Z}_2''$ is the diagonal subgroup of the centre Z. This group has centre and fundamental group $\cong \mathbb{Z}_2$, and its proper representations are the tensor reps, that is the conjugacy classes $(0), (v)$. We note in passing that $\mathbb{Z}_2^{(\mathrm{diag})}$ is generated by (v).

4. $PSO(32) = \mathrm{Spin}(32)/\mathbb{Z}_2' \times \mathbb{Z}_2''$ with trivial centre $C = \{1\}$ and fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$. This is the projectivized special orthogonal group. Only reps in the class (0) are proper reps of this group.

The group of conjugacy classes also inherits a scalar product from Γ_W. We have seen above that scalar products between weights are determined modulo $\mathbb{Z}$ by their conjugacy class, while the norm squared of a weight is determined modulo $2\mathbb{Z}$ by their conjugacy class. For the conjugacy classes of D_n we find:

	(0)	(v)	(s)	(c)
(0)	0	0	0	0
(v)	$*$	0	$\frac{1}{2}$	$\frac{1}{2}$
(s)	$*$	$*$	$\frac{n}{4}$	$\frac{n-2}{4}$
(c)	$*$	$*$	$*$	$\frac{n}{4}$

and

$\lvert(0)\rvert^2$	$\lvert(v)\rvert^2$	$\lvert(s)\rvert^2$	$\lvert(c)\rvert^2$
0	1	$\frac{n}{4}$	$\frac{n}{4}$

We can draw several interesting conclusions from these tables:

1. The lattice $D_n^{(0),(v)}$ is integral but not even.

2. A lattice which contains (v) together with one of the spinor conjugacy classes cannot be integral.

3. A lattice which contains both spinor conjugacy classes cannot be even.

4. The lattices $D_n^{(0),(s)}$ and $D_n^{(0),(c)}$ are integral for $n = 4m$, $m \in \mathbb{Z}$ and integral and even for $n = 8m$.

Moreover, for $n = 8$ the lattices $D_n^{(0),(s)}$ and $D_n^{(0),(c)}$ are even and self-dual, and the weights $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots\right)$ of the Weyl spinor reps have norm-squared two. They extend the root system D_8 to the root system of a larger ADE Lie algebra, namely E_8. The root lattice of E_8 is $E_8^{(0)} = D_8^{(0),(s)}$. Note that E_8 only has one conjugacy class of representations, the class (0) of the adjoint. The lattice $E_8^{(0)} = D_8^{(0),(s)}$ is, up to isometry, the only eight-dimensional Euclidean even self-dual lattice.

Finally, consider the problem of construction Klein factors for the representations of ADE Lie algebras by vertex operators. Let

$$\tilde{E}^{\alpha}(z) = e^{i\alpha X(z)}$$

where $\alpha \in \Gamma$ and Γ is an even integral lattice. Then

$$\begin{aligned} \tilde{E}^{\alpha}(z)\tilde{E}^{\beta}(w) &= (z-w)^{\alpha\cdot\beta}\tilde{E}^{\alpha+\beta}(w) + \cdots \text{ for } |z| > |w| \\ \tilde{E}^{\beta}(w)\tilde{E}^{\alpha}(z) &= S(\alpha,\beta)(z-w)^{\alpha\cdot\beta}\tilde{E}^{\alpha+\beta}(w) + \cdots \text{ for } |w| > |z| \end{aligned}$$

where $S(\alpha, \beta) = (-1)^{\alpha\cdot\beta}$. Depending on $\alpha\cdot\beta$ being even or odd, the OPE implies commutation/anticommutation relations between the modes $\tilde{E}^{\alpha}_m = \oint \frac{dz}{2\pi i} z^m \tilde{E}^{\alpha}(z)$.

To obtain a standard 'bosonic' OPE with commutation relations between all modes, we need

$$\begin{aligned} E^{\alpha}(z)E^{\beta}(w) &= (z-w)^{\alpha\cdot\beta}\tilde{E}^{\alpha+\beta}(w)+\cdots \text{ for } |z|>|w| \\ E^{\beta}(w)E^{\alpha}(z) &= (z-w)^{\alpha\cdot\beta}\tilde{E}^{\alpha+\beta}(w)+\cdots \text{ for } |w|>|z| \end{aligned}$$

To obtain such $E^{\alpha}(z)$ from $\tilde{E}^{\alpha}(z)$ we need $\mathbb{Z}_2$ valued functions c_{α} such that

$$c_{\alpha}c_{\beta} = S(\alpha,\beta)c_{\beta}c_{\alpha}$$

so that we can set

$$E^{\alpha}(z) = \tilde{E}^{\alpha}(z)c_{\alpha}$$

This amounts to modifying the abelian group law for $(\Gamma,+)$, where $(\alpha,\beta)\mapsto \alpha+\beta=\beta+\alpha$ to a new, non-commutative law, where $c_{\alpha}c_{\beta}=\epsilon(\alpha,\beta)c_{\alpha+\beta}=S(\alpha,\beta)c_{\beta}c_{\alpha}$. The multiplication in the group $(\hat{\Gamma},\cdot)$ is defined by the $\mathbb{Z}_2=\{\pm 1\}$ valued function $\epsilon(\alpha,\beta)$.

This is a group extension problem. Given the group $\Gamma=(\Gamma,+)$ and the abelian group $A=(\mathbb{Z}_2,\cdot)$ we want to find a group $\hat{\Gamma}=(\hat{\Gamma},\cdot)$ such that A is a normal subgroup of $\hat{\Gamma}$ and $\Gamma=\hat{\Gamma}/A$. The group $\hat{\Gamma}$ is a central extension of Γ by the abelian group A. $\hat{\Gamma}$ projects onto $\Gamma=\hat{\Gamma}/A$:

$$\pi:\hat{\Gamma}\to\Gamma$$

We can choose a section, that is right inverse of the projection:

$$s:\Gamma\to\hat{\Gamma}\ ,\quad \pi\circ s=\mathrm{Id}_{\Gamma}$$

The image $s(\Gamma)\subset\hat{\Gamma}$ is a subset, but in general not a subgroup. The group structure of $\hat{\Gamma}$ is defined by

$$s(g)s(g')=\epsilon(g,g')s(gg')\ ,\quad \epsilon(g,g')\in A$$

(where we use generic, multiplicative notation for elements of Γ, as the general construction also applies to central extensions of non-abelian groups by abelian groups.) Associativity of the group law of $\hat{\Gamma}$ implies

$$\epsilon(g,g')\epsilon(gg',g'')=\epsilon(g',g'')\epsilon(g,g'g'')\Leftrightarrow(\delta\epsilon)(g,g',g'')=e$$

where e is the group unit, that is, ϵ is an A-valued 2-cocycle on G.

One can chow that by changing the section by a 1-cochain one can impose the normalization

$$\epsilon(g,e)=\epsilon(e,g)=\epsilon(e,e)=e$$

Central extensions of Γ by A are classified by elements of $H^2(\Gamma,A)$. See [12] for group/Lie algebra extensions and cohomology.

We return to the specific case where $\Gamma \cong (\mathbb{Z}^n, +)$ and $A \cong (\mathbb{Z}_2, \cdot)$, and use additive notation for Γ. We note that $\epsilon(w,v)^{-1} = \epsilon(w,v)$ and associativity implies

$$\epsilon(v,w)\epsilon(v+w,x) = \epsilon(w,x)$$

and normalization is

$$\epsilon(v,0) = \epsilon(0,w) = 1$$

The commutator in $\hat{\Gamma}$ is $S(v,w) = (-1)^{v\cdot w}$. It can be shown that one can choose a cocycle s.t.

$$\epsilon(v+w,x) = \epsilon(v,x)\epsilon(w,x)\ , \ \ \epsilon(v,w+x) = \epsilon(v,x)\epsilon(v,x)$$

For an ADE lattice generated by simple roots α_i one explicit solution of the extension problem is

$$\epsilon(\alpha_i,\alpha_j) = \begin{cases} -1 & \alpha_i \cdot \alpha_j = -1\ ,\ i<j \\ 0 & \text{else} \end{cases}$$

which can be extended to the lattice to give

$$\epsilon(v,w) = (-1)^{v*w}\ ,\ v*w = \sum_{j>i} v^i w^j \alpha_i \cdot \alpha_j$$

The corresponding Klein factors c_v can be realized on the Hilbert space by the cocycle operators

$$c_v = (-1)^{p*v}$$

where p is the momentum operator. See [6, 13, 14].

Alternatively, one may dispense with cocycle factors and modify the commutation relations for the zero modes of the string [15], see also [16].

For more on Lie algebra lattices and their role in string theory, see [6, 17].

6 Lecture 6: BRS symmetry, Worldsheet supersymmetry and Vertex Operators (5/12/2022)

Bosonization of general b,c systems. (b,c systems with $\epsilon = -1$ are bosonic. 'Bosonization' thus means replacing a first order system b,c by a second order system with a boson ϕ.) Let b,c be fields with conformal weights λ, $1-\lambda$ and statistics $\epsilon = \pm 1$. This system has background charge $Q = \epsilon(1-2\lambda)$ and central charge $c_{b,c} = \epsilon(1-3Q^2)$. The ghost number current is $j(z) =: b(z)c(z) :$ For $Q \neq 0$ we have an anomalous hermiticity relation $j_0^\dagger = -j_0 - Q$ and ghost number conservation $\langle q'|q\rangle \neq 0 \Rightarrow q' = -q - Q$.

Define the boson ϕ by integrating the ghost current:

$$ij(z) = \epsilon\partial\phi(z) \Rightarrow \phi(z) = \int ij(z)dz\ , \quad \phi(z)\phi(w) = \epsilon\log(z-w) + \cdots$$

The exponential $e^{iq\phi(z)}$ generates states $|q\rangle$ of ghost charge $|q\rangle$ and have conformal weight $h_\phi = \frac{1}{2}\epsilon q(q+Q)$. The central charge of the ϕ-system is

$$c_\phi = 1 - 3\epsilon Q^2 = \begin{cases} \epsilon(1-3Q^2) = c_{bc} & \epsilon = 1 \\ c_{bc} + 2 & \epsilon = -1 \ . \end{cases}$$

Thus fermionic ghosts can be bosonized directly $\phi \leftrightarrow (b,c)$ while for bosonic ghosts we need an auxiliary system of central charge $c' = -2$: $(b,c) \leftrightarrow \phi \oplus (c' = -2)$.

The energy momentum tensor of the ϕ system is

$$T^\phi = \epsilon \frac{1}{2} \left(: j^2(z) - Q\partial j(z) : \right)$$

and action

$$S[\phi] = -\frac{1}{8\pi} \int_\Sigma d^2 z \sqrt{h} \left(h^{\alpha\beta} \epsilon \partial_\alpha \phi \partial_\beta \phi + Q R_h \phi \right) \ .$$

Note the extra term related to the background charge Q.

Bosonization:

1. For $\epsilon = 1$:
$$b(z) = e^{-i\phi(z)} \ , \ c(z) = e^{i\phi(z)} \ .$$

2. For $\epsilon = -1$. Introduce auxiliary $c = -2$ system (η, ξ) with $\lambda' = 1$, $1 - \lambda' = 0$. Thus $\epsilon' = 1$, $Q' = \epsilon'(1 - 2\lambda') = -1$, $c' = -2$. Then
$$b(z) = e^{-i\phi(z)} \partial\xi(z) \ , \quad c(z) = e^{i\phi(z)} \eta(z)$$
Note that this does not involve the zero mode ξ_0 of $\xi(z)$.

   We can bosonize the $c = -2$ system:
$$\eta(z) = e^{-i\chi(z)} \ , \quad \xi(z) = e^{i\chi(z)} \ , \quad : \eta(z)\xi(z) := \partial\chi(z) \ , \quad \chi(z)\chi(w) = \log(z-w) + \cdots$$
Then
$$b(z) = e^{-i\phi(z)} e^{i\chi(z)} \partial\chi(z) \ , \quad (z) = e^{i\phi(z)} e^{-\chi(z)} \ .$$

Here is a summary of bc systems appearing in the RNS string and their bosonization:

bc fields	λ	$1-\lambda$	ϵ	Q	c	Boson
$\psi^\pm$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	c	H
b, c	2	-1	1	-3	-26	φ
β, γ	$\frac{3}{2}$	$-\frac{1}{2}$	-1	2	11	ϕ
η, ξ	1	0	1	-1	-2	χ

We now put all elements together and consider the WS CFT of the RNS/ superstring/ heterotic string:

X^μ	$c, b \simeq \varphi$
$\psi^\mu \simeq H^i$	$\beta, \gamma \simeq \phi, \chi$

The WS fermions ψ^μ are related to the string coordinates X^μ by WS supersymmetry. The ghosts c, b and superghosts β, γ are related to the dynamical fields X^μ, ψ^μ by BRS(T) symmetry.

BRS symmetry allows to characterise the physical states of a gauge theory as equivalence classes of a cohomology of the gauge algebra.

Generically: For a (finite or infinite dimensional) Lie algebra of symmetries

$$[T_i, T_j] = f_{ij}^k T_k$$

introduce two sets of anticommuting parameters, ghosts c^i and antighosts b_i,

$$\{c^i, b_j\} = \delta^i_j$$

Then the BRS charge

$$Q_{\text{BRS}} = c^i \left(T_i + \frac{1}{2} T_i^{(\text{ghost})} \right) \text{ , where } T_i^{(\text{ghost})} = f_{ij}^k c^j b_k$$

is nilpotent and commutes with the Hamiltonian H:

$$Q_{\text{BRS}}^2 = \frac{1}{2}\{Q_{\text{BRS}}, Q_{\text{BRS}}\} = 0 \ , \ [Q_{\text{BRS}}, H] = 0 \ .$$

In the 'large' Fockspace, which includes the ghosts b, c along with the dynamical fields, a state $|\phi\rangle$ is physical if

$$Q_{\text{BRS}}|\phi\rangle = 0$$

and pure gauge (null, spurious) if

$$|\phi\rangle = Q_{\text{BRS}}|\psi\rangle$$

for some state $|\psi\rangle$. Thus gauge-inequivalent physical states are in one-to-one correspondence with cohomology classes of Q_{BRS}:

$$\mathcal{H} = \mathcal{F}/\simeq \ , \ |\phi\rangle \simeq |\phi'\rangle \Leftrightarrow |\phi\rangle - |\phi'\rangle = Q_{\text{BRS}}|\psi\rangle \ .$$

For the bosonic string where $\mathcal{F}$ is spanned by (X^μ, b, c) the BRS charge is

$$Q_{\text{BRS}} = \sum_m : c_{-m} \left(L_m^X + \frac{1}{2} L_m^{b,c} \right) :$$

where L_m^X and $L_m^{b,c}$ are the Virasoro operators for X^μ and the bc-system, respectively and where $c_{-m} = c^m$. The BRS charge is the integral of a BRS current

$$Q_{\mathrm{BRS}} = \oint \frac{dz}{2\pi i} j_{\mathrm{BRS}}$$

and its nilpotency is equivalent to the cancellation of central charge in the combined Virasoro algebra of the X, b, c system, generated by $L_m^{(\mathrm{total})} = L_m^X + L_m^{b,c}$:

$$Q_{\mathrm{BRS}}^2 = 0 \Leftrightarrow [L_m^{(\mathrm{total})}, L_n^{(\mathrm{total})}] = (m-n)L_{m+n}^{(\mathrm{total})} \Leftrightarrow c_{\mathrm{total}} = c_X + c_{b,c} = c_X - 26 = 0 \Leftrightarrow D = c_X = 26 \; .$$

The X, b, c system has an $sl(2, \mathbb{C})$ invariant ground state $|0\rangle$,

$$L_n|0\rangle = 0 \; , n \geq -1$$

which satisfies

$$p^\mu|\rangle = 0 \; , \; \alpha_m^\mu|0\rangle = 0 \; , m > 0$$

and

$$b_m|0\rangle = 0 \; , \; m \geq -1 \; , \quad c_n|0\rangle = 0 \; , \; n \geq 2 \; .$$

Since

$$c_1|0\rangle \neq 0 \; , \; c_0 c_1|0\rangle \neq 0$$

as well as $[L_0, c_1] = -c_1$, $[L_0, c_0] = 0$ these two states have lower energy (L_0 eigenvalue) then the $sl(2, \mathbb{C})$ vacuum, namely $L_{=}-1$. They have a ghost number (ghost charge, that is charge with respect to the ghost number current $j_{gh}(z) =: b(z)c(z) :$) of $1, 2$, respectively.

Due to the ghost number anomaly, the two sates are adjoint to each other. Remember that

$$\langle q'|q\rangle \neq 0 \Leftrightarrow q' = -q - Q$$

which for $Q = -3$ requires $q' = -q + 3$, e.g. $q' = 2, q = 1$ (or $q' = 0, q = 3$).

The states

$$|0\rangle_T = c_1|0\rangle = c(0)|0\rangle \; , \quad {}_T\langle 0| = \langle 0|c_{-1}c_0$$

are sometimes called the Tachyon vacuum and the adjoint Tachyon vacuum (where the adjoint takes into account the ghost number anomaly).

The Tachyon vacuum is physical, while the $\mathfrak{sl}(2, \mathbb{C})$ vacuum is not:

$$Q_{\mathrm{BRS}}|0\rangle_T = 0 \; , \quad Q_{\mathrm{BRS}}|0\rangle \neq 0 \; .$$

Moreover, the Tachyon vacuum is annihilated by all positive ghost modes

$$c_n|0\rangle_T = 0 \; , \quad b_n|0\rangle_T \; , \; n > 0 \; .$$

On the large Fock space, amplitudes are computed with respect to the Tachyon vacuum, equivalently with respect to the $\mathfrak{sl}(2, \mathbb{C})$ vacuum with the insertion of three ghost modes

$${}_T\langle 0| \ldots |0\rangle_T = \langle 0|c_{-1}c_0c_1 \ldots |0\rangle$$

The ghost modes c_{-1}, c_0, c_1 are the only ones which neither annihilate $|0\rangle$ nor $\langle 0|$. As fermionic gauge parameters they a paired with the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra of the Virasoro algebra, which exponentiates (on the WS sphere) to the group of globally well defined finite conformal transformations. In the path integral context this means that the path integral over ghosts b, c has three zero modes. Without the insertion of c_{-1}, c_0, c_1 the integrals $\int dc_{-1}dc_0dc_1\cdots$ over these zero modes would result in the path integral being zero. This is an effect similar to the overcounting in the integral over WS metrics.

On the large Fock space, physical states take the form

$$|\phi\rangle_X \otimes |0\rangle_T$$

where we write X-part and b, c-part as separate factors. BRS invariance implies that $|\phi\rangle$ satisfies

$$(L_0^X - 1)|\phi\rangle_X = 0\ , \quad L_m^X|\phi\rangle_X = 0\ , m > 0$$

The vertex operator V_ϕ generating $|\phi\rangle_X$ must have conformal weight $h_\phi = 1$.

The bosonic string with dynamical fields X^μ is extended to fermionic string or RNS string by adding worldsheet fermions ψ^μ, which have conformal weight 1/2. Each real fermion (Euclidean continuation of a Minkwoski signature Majorana spinor) contributes 1/2 to the central charge. The fermions have an energy momentum tensor $T_\psi(z) \propto :\psi(z)\partial\psi(z):$ (for one chiral sector) which combines with energy momentum tensor of the bosons X^μ to the total ('matter') energy momentum tensor (here matter means leaving out the ghosts). In D-dimensions the RNS string (X^μ, ψ^μ) has central charge $3D/2$.

The ψ-sector of the Fock space is generated by applying fermionic creation operators to the NS/R ground state satisfying

$$b_r^\mu|0\rangle = 0\ , \quad r > 0\ (NS)\ ,\ d_m^\mu|0\rangle = 0\ , \quad m > 0\ (R)\ .$$

In the combined X^μ, ψ^μ system physical states satisfy

$$(L_0 - a)|\phi\rangle = 0\ , \quad L_m|\phi\rangle = 0\ , m > 0$$

where

$$a = \begin{cases} \frac{1}{2} & (NS) \\ 0 & (R) \end{cases}$$

Since $[L_0, d_0^\mu] = 0$, the application of Ramond sector ψ^μ zero modes does not change the mass of the state. In particular the R-ground state is degenerate and carries a representation of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\ , \gamma^\mu := \sqrt{2}d_0^\mu\ .$$

In even dimensions, the Clifford algebra has a unique irreducible representation, which has dimension $2^{D/2}$. This representation can be constructed by taking the linear combinations

$$\psi_I^\pm = \left\{ \begin{array}{ll} \psi_0^\pm = \frac{i}{2}\left(\gamma^0 \pm \gamma^1\right) & \\ \psi_i^\pm = \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right) & i = 1, \ldots \frac{D-2}{2} \end{array} \right\}, I = 0, \ldots \frac{D-2}{2} = n - 1$$

(where $n = D/2$), and observing that $\psi_I^{\pm}$ are fermionic creation and annihilation operators:

$$\{\psi_I^+, \psi_J^-\} = 0 \ .$$

Then by starting from a highest way state

$$\psi_I^+|0\rangle = 0$$

the representation is spanned by

$$(\psi_0^-)^{m_0} \cdots (\psi_{n-1}^-)^{m_{n-1}}|0\rangle \quad m_k \in \{0, 1\}$$

This representation clearly has dimension $2^n = 2^{D/2}$. By restriction, the Clifford representation becomes a Spin representation, which is reducible and decomposes into Weyl spinor representations of dimension 2^{n-1}. States can be labeled by the weights of the corresponding $D_n \cong \mathfrak{so}(2n)$ representations:

$$(\psi_0^-)^{m_0} \cdots (\psi_{n-1}^-)^{m_{n-1}} = \left| \pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2} \right\rangle$$

where the sign of the i-th term is +1/-1 if $m_{i-1} = 0, 1$. We can also use a notation where R-ground states are labeled by indices $a = 1 \ldots, 2^n = 2^{D/2}$ and γ^μ are represented by γ-matrices:

$$\gamma^\mu \ : |a\rangle \mapsto (\gamma^\mu)^a{}_b |b\rangle \ .$$

A system of D free bosons and D free fermions exhibits worldsheet supersymmetry with supercurrent

$$T_F(z) = \frac{i}{2} : \psi^\mu(z) \partial X(z) :$$

which transforms ∂X^μ into ψ^μ and vice versa. The (total, matter) energy momentum tenor $T(z) = T_X(z) + T_\psi(z)$ and the supercurrent form a close algebra under OPE, called the $N = 1$ superconformal algebra. The supercurrent carries conformal weight $h_{T_F} = 3/2$. The modes of the supercurrent are denoted $F_m, m \in \mathbb{Z}$ for the NS and $G_r, r \in \mathbb{Z}+\frac{1}{2}$ for the R-sector. $G_{-1/2}$ is the worldsheet supercharge which squares to the worldsheet translation operator $G_{-1/2}^2 = L_{-1}$. The remaining physical state conditions for the X^μ, ψ^μ system are:

$$F_m|\phi\rangle = 0 \ , \quad m > 0(NS) \ , \quad G_r|\phi\rangle = 0 \ , \quad r > 0(R)$$

In the BRS description the theory is augmented by ghosts b, c which are anticommuting tensor fields, and superghosts β, γ which are commuting spinor fields. One then forms an energy momentum tensor for the combined 'matter' X, ψ and (super-)ghost b, c, β, γ system. Since $c_{b,c} = -26$ and $c_{\beta,\gamma} = 11$ the conformal anomaly cancels for

$$c_X + c_\psi + c_{b,c} + c_{\beta,\gamma} = \frac{3}{2}D - 15 = 0 \Rightarrow D = 10 \ .$$

This also is the condition for nilpotency of the BRS formed out of the $N = 1$ SCA and the b, c, β, γ ghosts.

Similarly to the b, c system, the β, γ system has positive modes which do not annihiliate the $\mathfrak{sl}(2, \mathbb{C})$ vacuum. The condition the $\beta(z), \gamma(z)$ create regular states at $z = 0$ implies

$$\beta_m|0\rangle = 0 \ , m > -\frac{3}{2} \ , \quad \gamma_n|0\rangle = 0 \ , n \geq \frac{3}{2} \ ,$$

where $m, n \in \mathbb{Z}$ in the NS sector and $m, n \in \mathbb{Z} + \frac{1}{2}$ in the R sector. The operators $\gamma_{1/2}$ and γ_1 do not annihilate the $\mathfrak{sl}(2, \mathbb{C})$ vacuum and lower the L_0 eigenvalue. Since γ_m are bosonic operators, one can apply any number of such operators and lower the L_0 eigenvalue without any lower bound. However, the β, γ ghosts are not dynamical fields, and this does not signal an instability, but rather an infinite degeneracy in the realisation of physical states. This is referred to as (super-)ghost pictures or pictures for short. From the previous discussion of general b, c systems we know that transition amplitudes between states of different ghost number are non-zero if the ghost charges differ by the background charge of the system

$$\langle q'|q\rangle \neq 0 \Rightarrow q' = -q - Q$$

For the superghosts β, γ, $Q = 2$.

For a general b, c system states $|q\rangle$ with the properties

$$b_n|q\rangle n >= 0, \ n > \epsilon q - \lambda \ , \quad c_n|q\rangle = 0, n \geq -\epsilon q + \lambda$$

are called q-vacua, and a choice of q selects the respective ghost/superghost picture relative to which one defines the states. While for the b, c system physical states take the same form in all pictures (so that picture changing is not much of an issue), physical states, i.p. vertex operators look differently in different superghost pictures, and 'picture changing' (that is the isomorphism between representations of states relative to different pictures) is a non-trivial map. Moreover, for the β, γ sector one cannot avoid to work in more than one picture in order to cover all possible amplitudes. States of ghost charge q can be generated from the $\mathfrak{sl}(2, \mathbb{C})$ vacuum by application of the operator $e^{iq\phi}$,

$$|q\rangle = e^{iq\phi(0)}|0\rangle$$

where ϕ is scalar field which bosonizes the b, c system. For cases with $\epsilon = -1$, like β, γ, where the ghosts are themselves bosons, 'bosonization' means to transform a first order system into a second order system.

To each physical state one can associate a vertex operator. Momentum eigenstates:

$$|k\rangle \longleftrightarrow : e^{ik_\mu X^\mu(z,\bar{z})} :$$

Bosonic excitations (one chiral sector only)

$$\alpha^\mu_{-m}|k\rangle \longleftrightarrow : \partial^m X^\mu(z) e^{ik_\nu X^\nu(z)} :$$

Fermionic excitations, NS-sector, bosonized description:

$$b^{\mu_1}_{-r_1} \cdots |k\rangle \longleftrightarrow : e^{v\cdot H(z)} e^{ik\cdot X(z)} :$$

where $v \in D_5^{(0),(v)}$ and $H = (H_1, \dots, H_5)$. Fermionic excitations, R-sector, bosonized description:

$$d^{\mu_1}_{-m_1} \cdots |k\rangle \longleftrightarrow : e^{v\cdot H(z)} e^{ik\cdot X(z)} :$$

where $v \in D_5^{(s),(c)}$. I.p. the R-ground states are generatred by the so-called spin field

$$|\alpha\rangle = S_\alpha(0)|0\rangle = e^{i\alpha\cdot H(z)}|0\rangle \ , \ \alpha = \left(\pm\frac{1}{2}, \dots \pm \frac{1}{2}\right)$$

The physical ground states $\alpha, k\rangle$ of the R-sector correspond to massless particles in the reducible $2^{n-1} \oplus 2^{n-1}$ (spinor$_+$ $\oplus$ spinor$_-$) representation. The natural candidate for a vertex operator is

$$: e^{i\alpha\cdot H} e^{ik\cdot X} :$$

This operator has conformal weight

$$h = \frac{1}{2}\alpha^2 + \frac{1}{2}k^2 = \frac{5}{8} + \frac{1}{2}k^2$$

Since the conformal weight for a physical vertex operator must be $h = 1$, the mass of this state is

$$\frac{5}{8} + \frac{1}{2}k^2 = 1 \Rightarrow M^2 = -k^2 = -2(1 - \frac{5}{8}) \neq 0$$

This can be resolved within the BRS framework. As we have seen, R-sector states carry superghost charge $q \in \mathbb{Z}+\frac{1}{2}$. Since the ghost superghost charge cannot be zero, vertex operators for space-time fermions must contain a superghost operator. If we include the operator which generates superghost charge $-1/2$ we obtain the fermion vertex operator

$$V_{-1/2} = u^\alpha : S_\alpha e^{-\frac{i}{2}\phi} e^{ik\cdot X}$$

Since $e^{-\frac{i}{2}\phi}$ has conformal weight

$$\frac{1}{2}\epsilon q(q+Q) = \frac{1}{2}(-1)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right) = \frac{3}{8}$$

the vertex operator $V_{-1/2}$ has conformal weight 1 for $k^2 = 0$, (massless state).

In the BRS framework, physical states on the large Fock space are BRS-closed, and the corresponding (unintegrated) vertex operators commute with the BRS charge up to a total derivative

$$[Q_{\mathrm{BRS}}, V_{\mathrm{phys}}] = \partial(\cdots) \ .$$

For $V_{-1/2}$ this imposes the condition that the spinor u^α satisfies the massless, Fourier transformed Dirac equation:

$$(k_\mu \gamma^\mu)^\alpha{}_\beta u^\beta = 0 \, .$$

Correlation functions can only be non-zero if the background superghost charge is compensated (we assume that the ghost charge has been compensated by the insertion of $c_{-1}c_0c_1$). Using $V_{-1/2}$ we can form the non-zero correlation function

$$\langle V_{-1/2} V_{-1/2} V_{-1/2} V_{-1/2} \rangle \neq 0$$

but to construct general non-zero correlation functions involving the R-groundstate we need versions of the vertex operator which carry different ghost charges. Versions of vertex operators which carry different superghost charge are said to belong to different (super-)ghost pictures, or pictures for short. The map which relates a vertex operator in the picture with charge q to an equivalent vertex operator in the picture with charge $q+1$ is called picture changing (this is invertible). The picture changing operator P_+ can be realized as a combination of the worldsheet supercorrent and the operator increasing ghost charge by one unit:

$$P_+ = \oint \frac{dz}{2\pi i} e^{i\phi} \psi^\mu \partial X_\mu$$

Literature: [6]

End of Part I

I plan to continue lectures in 2023, as well as keep polishing these notes. Some material that I have already prepared is in the appendix.

A Further material

Vertex operators in the bosonized formulation take the form

$$V =: \partial^N X^\mu e^{i\lambda\cdot H + iq\phi} e^{ik\cdot X} :$$

where

$$\begin{array}{lll} \lambda \in D_5^{(0),(v)} & q \in \mathbb{Z} & (NS) \\ \lambda \in D_5^{(s),(c)} & q \in \mathbb{Z} + \frac{1}{2} & (R) \end{array}$$

If physical, such an operator generates a state of mass

$$\alpha' M^2 = 4\left(\frac{1}{2}\lambda^2 + N - 1 - \frac{1}{2}q(q+2)\right)$$

with ghost charge $N_{gh} = 0$ and superghost charge $N_{sgh} = q$. Since operators $: \exp(i\lambda \cdot H + iq\phi) :$ have OPEs of the form

$$: e^{i(\lambda\cdot H + iq\phi} :: e^{i(\lambda'\cdot H + iq'\phi} := (z-w)^{\lambda\cdot\lambda' - qq'} : e^{i(\lambda+\lambda')\cdot H + i(q+q')\phi} + \cdots$$

we are led to defining the non-Euclidean Lie algebra lattice $D_{5,1}$;

$$w = (\lambda, q) \in D_{5,1}\ , \quad w \cdot w' = \lambda \cdot \lambda' - qq'$$

We also define the lattice $D_1 = \frac{1}{2}\mathbb{Z}$, so that $D_{5,1} = (D_5 D_1)_{cc}$, where cc is a list which tells us which conjugacy classes of D_5 and D_1 we combine to obtain the $D_{5,1}$ lattice.

The definition of $D_{5,1}$ is useful, because it captures physical properties of the vertex operator algebra in terms of lattice properties. I.p. the vertex operator algebra is local, that is, OPEs don't have branch cuts, if $w \cdot w' \in \mathbb{Z}$, that is, if the lattice $D_{5,1}$ is *integral*. It is also possible to pick representatives for (gauge fixed) physical states by a version of the LC gauge. Decompose

$$w = (u, x) \in D_{5,1}\ \text{, where } u \in D_4 \simeq \mathfrak{so}(8)$$

where $\mathfrak{so}(8)$ is the transverse rotation subgroup of the Lorentz group $\mathfrak{so}(1,9)$ and where x is fixed to be

$$x = \begin{cases} (0,-1) & (NS) \\ \left(-\frac{1}{2}, -\frac{1}{2}\right) & (R) \end{cases}$$

Thus we choose representatives with superghost charge -1 for the NS-sector and $-1/2$ for the R-sector. We also define a lattice $D_{1,1}$ which is like D_2 but with indefinite bilinear form. Thus $x \in D_{1,1}^{(v)}$ for NS states and $x \in D_{1,1}^{(s)}$ for R states. The mass formula is

$$\alpha' M^2 = 4\left(\frac{1}{2}u^2 + N - \frac{1}{2}\right) = 4\left(\frac{1}{2}w^2 + N - 1 + q\right)$$

Let us list the states corresponding to the lowest mass levels:

$w = (u,x)$	$D_{5,1}$	D_4	$D_{1,1}$	$\mathfrak{so}(8)$	$\alpha' M^2$
$(0 \cdots 0 \vert 0, -1)$	(v)	(0)	(v)	1	-2
$(\cdots \pm 1 \cdots \vert 0, -1)$	(0)	(v)	(v)	8_V	0
$\left(\pm\frac{1}{2} \cdots \vert -\frac{1}{2}, -\frac{1}{2}\right)$	(s)	(s)	(s)	8_S	0
	(c)	(c)	(s)	8_C	0
$(\pm 1 \cdots \pm 1 \cdots \vert 0, -1)$	(v)	(0)	(v)	28_{adj}	2

The RNS string is not modular invariant. The modular invariant type-II superstring theories are obtained by the GSO projection. In the above 'covariant lattice' description the GSO projection eliminates all mass levels $\alpha' M^2 = -2, 2, 6, \ldots$ and half of the R-states at the mass levels $\alpha' M^2 = 0, 4, 8, \ldots$. At the massless level, one of the spinor representations, say 8_C is eliminated leaving a massless ground states $8_V + 8_S$. This is the on-shell content of ten-dimensional $N = 1$ vector supermultiplet (one vector boson, on Majorana-Weyl fermion). It can be shown that the NS and R sector partition functions match level by level, that is, type-II superstrings have the same number of (spacetime) bosonic and fermionic states at each mass level (Jacobi's 'equatio identical satis abstrusa'). The bosonic or covariant lattice formulation allows to construct the spacetime supercharges explicitly.

Part of the motivation for the GSO projection can be read of from the scalar products between $D_{5,1}$ conjugacy classes:

	(0)	(v)	(s)	(c)
(0)	0	0	0	0
(v)		0	$\frac{1}{2}$	$\frac{1}{2}$
(s)			0	$\frac{1}{2}$
(c)				0

This shows that any theory which contains the combinations $(v), (s)$ or $(v), (c)$ or $(s), (c)$ of $D_{5,1}$ conjugacy classes is not local, because the corresponding lattice is not integral. However the combinations $(0), (s)$ and $(0), (c)$ define integral lattices. The two type-II superstring theories are based on the lattices

$$(D_{5,1}^{(0),(s)})_L \oplus \begin{cases} (D_{5,1}^{(0),(c)})_R & (IIA) \\ (D_{5,1}^{(0),(s)})_R & (IIB) \end{cases}$$

while the two heterotic string theories are based on the lattices

$$\Gamma_{16;5,1} = (\Gamma_{16})_L \oplus (D_{5,1}^{(0),(s)})_R\ ,\ \ \Gamma_{16} = \begin{cases} E_8 E_8 & (HE) \\ D_{16}^{(0),(s)} & (HO) \end{cases}$$

Heterotic vertex operators contain bosonic exponentials of the form

$$V = e^{i w_L \cdot X_L + i\lambda \cdot H + iq\phi}$$

where

$$w = (w_L, \lambda, q) \in \Gamma_{16;5,1}$$

with scalar product (as appearing in the OPE)

$$w \cdot w' = -w_L \cdot w'_L + \lambda \cdot \lambda' - qq'$$

Thus the scalar product associated with the heterotic lattice $\Gamma_{16;5,1}$ has signature $(-)^{16}(+)^5(-)^1$.

Toroidal compactifications to even dimensions $d = 10 - 2n$ are described by lattices of the form

$$\Gamma_{16+2n,2n} \oplus D_{5,1}^{(0),(s)}$$

where the Narain lattice $\Gamma_{16+2n,2n}$ encodes the ten-dimensional gauge charges together with momenta and windings associated with a torus T^{2n}. This lattice can be re-arranged to the form

$$(\Gamma_{16+2n})_L \oplus (\Gamma_{3n} + D_{5-n,1})_R$$

with signature $(-)^{16+2n}(+)^{5+2n}(-)^{1}$.

Here Γ_{16+2n} encodes the ten-dimensional gauge charges and the $2n$ left-moving momenta, Γ_{3n} the $2n$ right-moving momenta and $2n$ right-moving ws fermions and $D_{5-n,1}$ encodes the d-dimensional Lorentz Lie algebra $\mathfrak{so}(10-2n-1,1)$ together with the ghost number. In the LC gauge it is decomposed into $D_{4-n}D_{1,1}$, where $D_{4-2n} \cong \mathfrak{so}(8-2n)$ encodes the transverse (=physical) degrees of freedom and $D_{1,1}$ the longitudinal/timelike (= pure gauge oscillations and the ghosts). This lattice is integral, but not even: ST fermionic states correspond to lattice with odd square norm. An equivalent formulation is obtained by the so called lattice map which replaces (conjugacy class by conjugacy class) the 'ghost lattice' $D_{1,1}$ by D_4. The resulting lattice $\Gamma_{16+2n,8+2n}$ is even with signature $(-)^{16+2n}(+)^{8+2n}$. The LC gauge now takes the form

$$w = (u|y) \in D_{4-n}D_4 \ , \ \ w \in D_{4-n} \cong \mathfrak{so}(8-2n) \ , \ y = \begin{cases} (0,0,0,0) & (NS) \\ (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) & (R) \end{cases}$$

In general, replacing a D-type lattice D_n by a lattice D_{n+8p} $p \in \mathbb{Z}$, preserves all scalar products modulo 1 and square norms modulo 2. When shifting by 4 units rather than 8 even lattices are mapped to odd lattices, and vice versa. This map can be extended to include indefinite lattices: $D_{1,1} \mapsto D_4$, $D_{2,1} \mapsto D_5$, $D_{3,1} \mapsto D_6$, ..., $D_{5,1} \mapsto D_8$, ...

For ten-dimensional heterotic strings it can be shown that the theory is local and modular invariant if the lattice $\Gamma_{16,8} = \Gamma_{16}D_8$ is even and self-dual. Since one can change the signature modulo 8 without changing one scalar products modulo 1 and square norms modulo 2, this is equivalent the condition that the Euclidean lattice $\Gamma_{24} = \Gamma_{16}D_8$ is even self-dual. Up to $O(24)$ transformations, there are 24 even selfdual Euclidean lattices in dimension 24. Only those which take the form $\Gamma_{16}D_8$ define ten-dimensional heterotic string theories. This leaves 8 lattices, defining 8 ten-dimensional heterotic string theories which can be realized using free bosons.[4] It can further be shown that the theory has ten-dimensional $N=1$ supersymmetry if the D_8 factor extends to E_8 inside Γ_{24}, is non-supersymmetry and tachyon-free if D_8 is not embedded into a larger Lie algebra lattice and has tachyons if D_8 is embedded into a larger D_n lattice:

$(D_8)_L$	Susy	Tachyons
$D_8 \subset E_8$	$N=1$	No
D_8	$N=0$	No
$D_8 \subset D_m$	$N=0$	Yes

As a result there are 2 supersymmetric, 1 non-supersymmetric tachyon-free and

[4]There exists a tachyonic ten-dimensional heterotic string theory with gauge group E_8 which cannot be expressed in terms of free bosons. It has an E_8 current algebra with level $k=2$.

5 non-supersymmetry tachyonic heterotic string theories in ten dimensions

Lattice	Susy	Tachyons	Gauge group
$E_8E_8E_8$	$N=1$	No	$E_8 \times E_8$
$D_{16}E_8$	$N=1$	No	$\mathrm{Spin}(32)/\mathbb{Z}_2$
$D_8D_8D_8$	$N=0$	No	$SO(16) \times SO(16)$
D_{24}	$N=0$	Yes	$SO(32)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Four-dimensional heterotic string theories can be defined by even selfdual lattices of the form $\Gamma_{22,14} = (\Gamma_{22})_L + (\Gamma_9 D_5)_R$ where $D_5 = D_1 D_4$, $D_1 \cong \mathfrak{so}(2)$ encodes the rep of the transverse rotation group, D_4 is the ghost lattice. It can be shown that such a theory has $N = 1, 2, 4$ four-dimensional spacetime supersymmetry if the D_5-factor extends to an exceptional Lie algebra lattice, $D_5 \subset E_6, E_7, E_8$.

In dimensions, the spacetime supercharges take the following form in the $-1/2$ picture:

$$Q_\alpha = \oint \frac{dz}{2\pi i} e^{-\frac{i}{2}\phi} S_\alpha(z)$$

If this is a physical operator, the covariant lattice must contain vectors of the form $0, (s)$, where (s) are the spinor weights of D_8. These extend the root lattice of D_8 to the root lattice of E_8. This explains the connection between spacetime supersymmetry and the extension $D_8 \subset E_8$. (Similarly one can show that extensions $D_8 \subset D_m$ always imply that the spectrum contains tachyons.)

To construct the four-dimensional supercharges, we split the worldsheet CFT as $c_L = 26 = 4 + 22$ and $c_R = 15 = 6 + 9$. In the right chiral sector, we bosonize the four worldsheet fermions ψ^μ into two bosons $\varphi = (\varphi_1, \varphi_2)$. The currents for four-dimensional supercharges in the $-1/2$ picture must take the form

$$Q_\alpha(z) =: e^{-\frac{i}{2}\phi(z)} S_\alpha(z) \Sigma(z) :$$

where S_α is the four-dimensional spin field, and Σ is a field formed out of of the internal $c = 9$ CFT_R which contributes to the conformal weight such that $Q_\alpha(z)$ has weight one. It can be shown that Σ must take the form

$$\Sigma(z) = e^{i\frac{\sqrt{3}}{2}H(z)}$$

where $H(z)$ is a free boson. There is an associated conserved current $J(z) = i\sqrt{3}\partial H(z)$, and the internal CFT_R splits into a $U(1)$ current algebra carrying $c = 1$ and a remaining piece with central charge $c = 8$ which remains arbitrary, except that the $c = 9$ internal CFT_R is an $N = 1$ SCFT with $N = 1$ SCA generated by its energy momentum tensor $T(z)$ and supercurrent $T_F(z)$. The existence of the $U(1)$ current $J(z)$ implies that the OPE $J(z)T_F(w)$ contains a second supercurrent $\tilde{T}_F(z)$, that is, $(T(z), \tilde{T}_F(z))$ generate another $N = 1$

SCA. Moreover, the linear combinations $T_F^{\pm} = T_F \pm \tilde{T}_F$ carry charge ± 1 under the current J. As a result the fields $(T, T_F, \tilde{T}_F, J) \leftrightarrow (T, T_F^{\pm}, J)$ generate an extended $N = 2$ SCA. Thus four-dimensional spacetime supersymmetry implies extended worldsheet supersymmetry in the internal CFT. Conversely extended WS susy is necessary but not sufficient for ST susy. For a general heterotic compactification, the left-moving internal CFT will not have WS supersymmetry, so that overall we have $(0, 2)$ supersymmetry. There is a special subclass of heterotic models were the internal CFT is left/right symmetric so that one has $(2, 2)$ worldsheet supersymmetry. This does not lead to further enhancement of spacetime supersymmetry. For heterotic Calabi-Yau compactifications this is related to how anomaly cancellation is realized. $(2, 2)$ ws supersymmetry is the case of 'standard embedding' where the heterotic anomaly is cancelled by setting the spin connection on the internal sixfold equal to the $E_8 \times E_8$ gauge connection. These models are easier to analyse then those with generic, non-standard embeddings, which correspond to $(0, 2)$ ws susy.

The contribution from worldsheet fermions ψ^μ, $\mu = 0, \ldots, 3$, the internal $U(1)$ current algebra and the superghosts to vertex operators takes the form

$$V = e^{i\lambda \cdot \varphi} e^{i \frac{Q}{\sqrt{3}} H} e^{iq\phi}$$

This part of the CFT is therefore encoded by lattice vectors

$$p = \left(\frac{Q}{\sqrt{3}}, \lambda, q \right) \in u(1) + D_{2,1} = \Gamma_{3,1} \subset (\Gamma_{3n} D_{2,1})_R$$

where $u(1)$ is the lattice of $u(1)$ charges. By a lattice map, which preserves the scalar products between conjugacy classes, this lattice can be replaced:

$$\Gamma_{3,1} = u(1) + D_{2,1} \mapsto u(1) + D_5 = \Gamma_6$$

The sufficient condition for four-dimensional ST supersymmetry is that in addition to enhanced $N = 2$ WS susy in the internal CFT, the superghost charge and the $U(1)$ charge of physical states are correlated: either both of them are integer, $q, Q \in \mathbb{Z}$ or both are half-integer $q, Q \in \mathbb{Z} + \frac{1}{2}$. This correlation between q and Q is equivalent to the statement that Γ_6 is the E_6 lattice. Thus $N = 1$ ST supersymmetry is equivalent to $N = 2$ WS supersymmetry together with an E_6 current algebra (or $E_{3,1}$ current algebra, if we reverse the lattice map).

Similarly, $N = 2$ ST susy implies that the $c = 9$ CFT_R contains an $c = 3$ sector generated by two free bosons and two free fermions. The conditions on lattice vectors imply that the D_5 lattice extends into the E_7 lattice. A geometric realization are compactifications on $K3 \times T^2$. Finally, $N = 4$ ST susy implies that the internal $c = 9$ CFT_R consists of six free bosons and six free fermions. The D_5 lattice extends into an E_8 lattice and the geometric realization is by compactification on T^6.

To summarize, for four-dimensional heterotic strings space-time supersymmetry is equivalent to $(0, 2)$ supersymmetry of the internal CFT together with an extension of the right D_5 sublattice:

Lattice	ST Supersymmetry
$D_5 \subset E_6$	$N = 1$
$D_5 \subset E_7$	$N = 2$
$D_5 \subset E_8$	$N = 4$

While we have focussed on the heterotic string, it is clear that for type-II superstring the same construction carried out in both chiral sectors symmetrically, leads to $N = 2, 4, 8$ ST susy in four dimensions.

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