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LOWER BOUNDS ON THE LARGEST INHOMOGENEOUS APPROXIMATION CONSTANT

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ABSTRACT. For a given irrational number α and a real number γ in $(0,1)$ one defines the two-sided inhomogeneous approximation constant

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n|||n\alpha-\gamma||,$$

and the case of worst inhomogeneous approximation for α

$$\rho(\alpha) := \sup_{\gamma\notin\{m+l\alpha\,:\,m,l\in\mathbb{Z}\}} M(\alpha,\gamma).$$

We are interested in lower bounds on $\rho(\alpha)$ in terms of $R := \liminf_{i\to\infty} a_i$, where the a_i are the partial quotients in the negative (i.e., the 'round-up') continued fraction expansion of α. We obtain bounds for any $R\geq 3$ which are best possible when R is even (and asymptotically precise when R is odd). In particular when $R\geq 3$

$$\rho(\alpha)\geq \frac{1}{6\sqrt{3}+8}=\frac{1}{18.3923\ldots},$$

and when $R\geq 4$, optimally,

$$\rho(\alpha)\geq \frac{1}{4\sqrt{3}+2}=\frac{1}{8.9282\ldots}$$

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1. Introduction

For an irrational number α and a real number γ, we define the two-sided inhomogeneous approximation constant by

$$M(\alpha,\gamma) := \liminf_{|n|\to\infty} |n|||n\alpha-\gamma||,$$

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where $||x||$ denotes the distance from x to the nearest integer. Plainly this reduces to the classical homogeneous problem $\gamma = 0$ if $\gamma = m + l\alpha$ for some $m, l \in \mathbb{Z}$. The homogeneous problem is well understood, with $M(\alpha, 0)$ readily determined from the usual regular continued fraction expansion of $\alpha = [a_0; a_1, a_2, \ldots]$,

$$M(\alpha, 0) = \frac{1}{\limsup_{i\to\infty} (a_i + [0; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots])} \le \frac{1}{\sqrt{5}},$$

leading naturally to bounds in terms of the largest partial quotients

$$\frac{1}{\sqrt{r^2 + 4r}} \le M(\alpha, 0) \le \frac{1}{\sqrt{r^2 + 4}}, \qquad r := \limsup_{i\to\infty} a_i,$$

with equality for $\alpha = [0; \overline{1, r}] = \frac{1}{2}(\sqrt{r^2 + 4r} - r)$ and $\alpha = [0; \overline{r}] = \frac{1}{2}(\sqrt{r^2 + 4} - r)$.

For any α, we define the worst inhomogeneous approximation

$$\rho(\alpha) := \sup_{\gamma \notin \{m + l\alpha \,:\, m, l \in \mathbb{Z}\}} M(\alpha, \gamma).$$

By contrast with the homogeneous case, the inhomogeneous constant $\rho(\alpha)$ will be affected by the smallest partial quotients

$$R := \liminf_{i\to\infty} a_i. \tag{1}$$

In subsequent sections the a_i in this definition will refer to the partial quotients in the negative 'round-up' continued fraction expansion rather than the regular expansion. From a well-known theorem of Minkowski we have

$$\rho(\alpha) \le \frac{1}{4}, \tag{2}$$

see, for example, [1, Chap. III] or [12, IV.9], Grace [5] giving examples with $R = \infty$ and $\rho(\alpha) = \frac{1}{4}$. We are interested here in the lower bound for $\rho(\alpha)$. Absolute bounds

$$\rho(\alpha) \ge C \tag{3}$$

have some history. Davenport [2] obtained (3) with $C = \frac{1}{128}$, Ennola [3]

$$C = \frac{1}{16 + 6\sqrt{6}} = \frac{1}{30.69\ldots},$$

and in [9] the absolute lower bound was improved to

$$C = \frac{(\sqrt{10} - 3)(7 - \sqrt{13})}{(31 - 2\sqrt{10} - 3\sqrt{13})} = \frac{1}{25.1592\ldots} \tag{4}$$

See Rockett and Szüsz [12] for a simpler proof with $C = \frac{1}{32}$. The smallest known value of $\rho(\alpha)$, and hence an upper bound on the optimal absolute lower bound C, is still an example of Pitman [11]

$$\rho\left(\frac{\sqrt{3122285}-1097}{1094}\right)=\frac{547}{4\sqrt{3122285}}=\frac{1}{12.9213\ldots}.$$

More generally, [9] obtains bounds of the form $\rho(\alpha)\geq C^*(R)$, where the a_i in (1) are the partial quotients in the nearest integer continued fraction of α and so $R\geq 2$ (giving (4) when $R=2$ and an improvement when $R\geq 3$). The bound comes by constructing a γ^* with $M(\alpha,\gamma^*)\geq C^*(R)$. The values for small R are given in (17) below and the asymptotic behavior in (21). The goal here is to improve these $R\geq 3$ bounds when the $a_i\geq 2$ in (1) are the partial quotients in the negative continued fraction expansion of α rather than the nearest integer expansion. Of course, when $R\geq 2$ in the regular expansion or $R\geq 3$ in the negative expansion, the expansion eventually becomes the nearest integer expansion (in the remaining cases, $R=1$ for the regular expansion and $R=2$ for the negative expansion, the value of R can be much larger, even infinite, if one uses the nearest integer expansion). From this point on the a_i in our definition of R will refer to the negative expansion and we will assume that $R\geq 3$.

2. Preliminaries

Different algorithms have been used for computing $M(\alpha,\gamma)$, see Komatsu [6]. In this paper we will follow the approach of [8], which showed how $M(\alpha,\gamma)$ can be expressed in terms of the negative continued fraction expansion of α and a corresponding α–expansion of $\gamma\notin\{m+l\alpha\ :\ m,l\in\mathbb{Z}\}$. We start by recalling some notations and results from [8]. Since $||m+x||=||x||$ for any integer m, we may assume that $\alpha,\gamma\in(0,1)$. For an $\alpha\in(0,1)$ we define the negative continued fraction expansion

$$\alpha=\cfrac{1}{a_1-\cfrac{1}{a_2-\cfrac{1}{a_3-\cdots}}}=:[0;a_1,a_2,a_3,\cdots]^-, \tag{5}$$

where the integers $a_i\geq 2$ are generated by the algorithm

$$\alpha_0:=\{\alpha\}=\alpha,\qquad a_{n+1}:=\left\lceil\frac{1}{\alpha_n}\right\rceil,\qquad \alpha_{n+1}:=\left\lceil\frac{1}{\alpha_n}\right\rceil-\frac{1}{\alpha_n},$$

and the corresponding convergents $\dfrac{p_n}{q_n} := [0; a_1, a_2, \ldots, a_n]^-$. Here the p_n and q_n are increasing sequences generated by

$$p_{n+1} := a_{n+1}p_n - p_{n-1}, \qquad p_0 = 0, \quad p_{-1} = -1,$$

$$q_{n+1} := a_{n+1}q_n - q_{n-1}, \qquad q_0 = 1, \quad q_{-1} = 0.$$

For the negative expansion the convergents p_n/q_n form an increasing sequence converging to α. Notice, increasing the size of any partial quotient decreases the size of α, that is, for any $c \geq 1$

$$[0; a_1, \ldots, a_{i-1}, a_i + c, \ldots]^- < [0; a_1, \ldots, a_{i-1}, a_i, \ldots]^-. \tag{6}$$

We define

$$\alpha_n := [0; a_{n+1}, a_{n+2}, \ldots]^-, \qquad \bar{\alpha}_n := [0; a_n, a_{n-1}, \ldots, a_1]^-, \qquad D_n := q_n\alpha - p_n,$$

so that

$$D_n = \alpha_0\alpha_1 \cdots \alpha_n = a_n D_{n-1} - D_{n-2}, \qquad q_n = (\bar{\alpha}_1\bar{\alpha}_2 \cdots \bar{\alpha}_n)^{-1}.$$

We observe that

$$(a_1 - 1)D_0 + \sum_{i=2}^{\infty}(a_i - 2)D_{i-1} = 1, \qquad p_{n+1}q_n - p_n q_{n+1} = 1. \tag{7}$$

For any real number $\gamma \in (0, 1)$, we generate the integers b_i by the algorithm

$$\gamma_0 := \{\gamma\} = \gamma, \qquad b_{i+1} := \left\lfloor \frac{\gamma_i}{\alpha_i} \right\rfloor, \qquad \gamma_{i+1} := \left\{ \frac{\gamma_i}{\alpha_i} \right\},$$

so that

$$\gamma = \sum_{i=1}^{n} b_i D_{i-1} + \gamma_n D_{n-1} = \sum_{i=1}^{\infty} b_i D_{i-1}$$

gives the unique expansion of γ of the form $\sum_{i=1}^{\infty} b_i D_{i-1}$, called the *$\alpha$–expansion* of γ, with the following properties [8]:

(1) $0 \leq b_i \leq a_i - 1$ for all i,

(2) the sequence $\{b_i\}_i$ does not contain a block of the form $b_s = a_s - 1$ for some s, with $b_j = a_j - 2$ for all $j > s$ or with $b_k = a_k - 1$ for some $k > s$ and $b_j = a_j - 2$ for all $k > j > s$.

We define the sequence of integers t_k by $b_k = \frac{1}{2}(a_k - 2 + t_k)$. Then

$$\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i)D_{i-1}, \tag{8}$$

and

$$d_k^- := \sum_{j=1}^{k} t_j \left(\frac{q_{j-1}}{q_k}\right) = t_k\bar{\alpha}_k + t_{k-1}\bar{\alpha}_k\bar{\alpha}_{k-1} + t_{k-2}\bar{\alpha}_k\bar{\alpha}_{k-1}\bar{\alpha}_{k-2} + \cdots,$$

$$d_k^+ := \sum_{j=k+1}^{\infty} t_j \left(\frac{D_{j-1}}{D_{k-1}}\right) = t_{k+1}\alpha_k + t_{k+2}\alpha_k\alpha_{k+1} + t_{k+3}\alpha_k\alpha_{k+1}\alpha_{k+2} + \cdots$$

Notice that t_k and a_k have the same parity, and $-(a_k - 2) \le t_k \le a_k$. It was observed in [8] that

$$-(1-\bar{\alpha}_k) \le d_k^- \le (1+\bar{\alpha}_k), \quad -(1-\alpha_k) \le d_k^+ \le (1+\alpha_k), \tag{9}$$

with $d_k^+ \ge 1 - \alpha_k$ (respectively $d_k^- \ge 1 - \bar{\alpha}_n$) if and only if the sequence $t_{k+1}, t_{k+2}, \ldots$ (respectively $t_k, t_{k-1}, \ldots$) has the form $t_j = a_j$ for some $j > k$ (respectively $j \le k$) with $t_i = a_i - 2$ for any $k < i < j$ (respectively $j < i \le k$). Note that $t_i = a_i$ if and only if $b_i = a_i - 1$. When only finitely many of the $b_i = a_i - 1$, it was shown in [8] that the sequence of best positive and negative inhomogeneous approximations lies amongst the

$$Q_k := \sum_{i=1}^{k} b_i q_{i-1}, \qquad Q_k + q_{k-1}, \qquad -(q_k - q_{k-1} - Q_k), \qquad -(q_k - Q_k).$$

We will work with the value of $|n| \parallel n\alpha - \gamma \parallel$ for these four values of n expressed in the symmetrical form $s_1(k), \ldots, s_4(k)$ of [8, Theorem 1].

Lemma 2.1. *If $\gamma \notin \{m + l\alpha \,:\, m, l \in \mathbb{Z}\}$ and the α–expansion of γ has $t_i = a_i$ at most finitely many times, then*

$$M(\alpha, \gamma) = \liminf_{k\to\infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\},$$

where

$$s_1(k) := \frac{1}{4}(1 - \bar{\alpha}_k + d_k^-)(1 - \alpha_k + d_k^+)/(1 - \bar{\alpha}_k\alpha_k),$$
$$s_2(k) := \frac{1}{4}(1 + \bar{\alpha}_k + d_k^-)(1 + \alpha_k - d_k^+)/(1 - \bar{\alpha}_k\alpha_k),$$
$$s_3(k) := \frac{1}{4}(1 - \bar{\alpha}_k - d_k^-)(1 - \alpha_k - d_k^+)/(1 - \bar{\alpha}_k\alpha_k),$$
$$s_4(k) := \frac{1}{4}(1 + \bar{\alpha}_k - d_k^-)(1 + \alpha_k + d_k^+)/(1 - \bar{\alpha}_k\alpha_k).$$

We set $R := \liminf_{i\to\infty} a_i$, where the a_i are now the partial quotients in the negative expansion (5).

In [8, Corollary 1] and [7] we gave the upper bounds,

$$\rho(\alpha) \le \frac{1}{4}\left(1 - \frac{1}{R}\right) \tag{10}$$

when $R \ge 4$ is even, and

$$\rho(\alpha) \le \frac{1}{4}\left(1 - \frac{1}{R}\right)\left(1 - \frac{1}{R^2}\right) = \frac{1}{4}\left(1 - \frac{1}{R} - \frac{1}{R^2} + \frac{1}{R^3}\right) \tag{11}$$

when $R \ge 3$ is odd. These are both best possible upper bounds on $\rho(\alpha)$ in terms of R, equalling

$$\lim_{N\to\infty} \rho\left([0; \overline{R, RN}]^-\right).$$

When $R = 2$ the Minkowski bound (2) cannot be improved (the examples of Grace have infinitely many two's in their negative expansions).

Our goal here is to obtain a lower bound for $\rho(\alpha)$ when $R \ge 3$. For this, we first construct a $\gamma^* \in (0,1)$ and then we use Lemma 2.1 to compute $M(\alpha, \gamma^*)$, which gives a lower bound $\rho(\alpha) \ge M(\alpha, \gamma^*)$.

3. Main results

Consider a real number $\gamma^* \in (0,1)$ which has the unique α–expansion

$$\gamma^* = \sum_{i=1}^{\infty} b_i D_{i-1} = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i) D_{i-1}, \tag{12}$$

where the sequence $\{t_i\}$ is given by

$$t_i = \begin{cases} 0 & \text{if } a_i \text{ is even,} \\ (-1)^{j+1} & \text{if } a_i \text{ is the } j\text{th odd partial quotient.} \end{cases}$$

Notice that any two nonzero consecutive t_i have opposite signs and hence

$$|d_k^-| \le \bar{\alpha}_k, \quad |d_k^+| \le \alpha_k, \quad d_k^- d_k^+ \le 0. \tag{13}$$

We define two numbers β and δ

$$\beta := [0; \overline{R_*}]^- = \frac{1}{2}\left(R_* - \sqrt{R_*^2 - 4}\right), \qquad \delta := [0; R_{**}, \overline{R_*}]^- = \frac{1}{R_{**} - \beta}, \tag{14}$$

where

$$R_*, \;\; R_{**} := \begin{cases} R, \;\; R+1 & \text{if } R \text{ is even,} \\ R+1, \;\; R & \text{if } R \text{ is odd.} \end{cases}$$

We set

$$C(R) := \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)}, \tag{15}$$

observing that if R is even

$$C(R) = \frac{1}{4}\left(\frac{R-2}{\sqrt{R^2-4}+1}\right),$$

and if R is odd

$$C(R) = \frac{1}{4}\left(\frac{2R-2-\sqrt{(R+1)^2-4}}{\sqrt{(R+1)^2-4}-1}\right).$$

The value of $M(\alpha,\gamma^*)$ gives us a lower bound for $\rho(\alpha)$.

Theorem 3.1. *Suppose that* (5) *gives the negative continued fraction expansion of* α *and* $R = \liminf_{i\to\infty} a_i \geq 3$*. Then, with* γ^* *as in* (12) *and* $C(R)$ *as in* (15) *we have*

$$\rho(\alpha) \geq M(\alpha,\gamma^*) \geq C(R). \tag{16}$$

In particular, when $R = 3$*,*

$$\rho(\alpha) \geq C(3) = \frac{1}{6\sqrt{3}+8} = \frac{1}{18.3923\ldots},$$

and when $R \geq 4$

$$\rho(\alpha) \geq C(4) = \frac{1}{4\sqrt{3}+2} = \frac{1}{8.9282\ldots}.$$

For $R \geq 3$ the value of $C(R)$ improves the lower bound $C^*(R)$ of [9, Theorem 4]:

R	$C^*(R)^{-1}$	$C(R)^{-1}$
2	25.1592...	–
3	20.4874...	18.3923...
4	9.3372...	8.9282...
5	8.2500...	7.9497...
6	6.8120...	6.6568...
7	6.4643...	6.3431...
8	5.9109...	5.8306...

(17)

Of course, if all the $a_i \geq 3$ in the negative expansion the negative and nearest integer continued fraction expansions coincide. That is, the lower bound $C^*(R)$, $R \geq 3$, actually applies to a much larger class of α than $C(R)$ (and so is not surprisingly smaller). Better bounds are also given in [9] when the nearest integer expansion coincides with the regular expansion. Notice that the bound $C(R)$ increases to $1/4$ as $R \to \infty$; in particular, from (10) or (11) and Theorem 3.1, when $R \geq 3$

$$\rho(\alpha) = \frac{1}{4} \text{ if and only if } R = \infty. \tag{18}$$

Fukasawa [4] showed that (18) holds without the $R \geq 3$ condition when using the nearest integer continued fraction expansion (the restriction is needed here since large partial quotients in the regular expansion will cause long strings of 2's in the negative expansion). We note the asymptotic behavior of $C(R)$; when $R \geq 4$ is even

$$C(R) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{5}{R^2} - \frac{E_1(R)}{R^3}\right), \quad 7.3268 < E_1(R) < 11, \tag{19}$$

and when $R \geq 3$ is odd

$$C(R) = \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} - \frac{E_2(R)}{R^3}\right), \quad 6.1279 < E_2(R) < 10. \tag{20}$$

For comparison, we note the [9] bounds

$$C^*(R) = \begin{cases} \frac{1}{4}\left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right), & \text{if } R \text{ is even,} \\ \frac{1}{4}\left(1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3})\right), & \text{if } R \text{ is odd,} \end{cases} \tag{21}$$

with this lower bound asymptotically optimal (and hence $C^*(R)$ inevitably smaller than $C(R)$) when R is even. When R is odd, it is not known whether the 3 in the $3/R^2$ term in (21) is optimal.

Our lower bound $C(R)$ for $\rho(\alpha)$ is optimal when R is even.

Theorem 3.2. *If α has negative continued fraction expansion of period*

$$R+1,\ \underbrace{R, \ldots, R}_{l \text{ times}}, \tag{22}$$

with $R \geq 4$ even, then $\rho(\alpha) \to C(R)$ as $l \to \infty$.

When $R \geq 3$ is odd, it will be clear from the proof of Theorem 3.1 that if α has negative continued fraction expansion of period

$$R, \underbrace{R+1, \ldots, R+1}_{l \text{ times}}, \tag{23}$$

then $M(\alpha, \gamma^*) \to C(R)$ as $l \to \infty$. So the bound $M(\alpha, \gamma^*) \geq C(R)$ in Theorem 3.1 is still best possible. However γ^* is no longer the best choice of γ; as we observe at the end of the paper, for $R \geq 5$ these α have

$$\lim_{l\to\infty} \rho(\alpha) = \frac{\left(1-2\delta+\frac{2\delta\beta}{1+\beta}\right)(1-\beta)}{4(1-\delta\beta)} = \frac{1}{4}\left(1-\frac{3}{R}+\frac{6}{R^2}+O\left(R^{-3}\right)\right). \tag{24}$$

We need a more complicated example to show the asymptotic sharpness of our lower bound when R is odd.

Theorem 3.3. *If R is odd and α has negative continued fraction expansion*

$$\alpha = [0; \overline{R, R, R+1, R, R+1, R+1, R, R+1, R+1, R, R+1}]^-,$$

then

$$\rho(\alpha) = \frac{1}{4}\left(1-\frac{3}{R}+\frac{4}{R^2}+O\left(R^{-3}\right)\right).$$

For $R = 3, 5$ and 7 the period two examples

$$\begin{aligned}
\rho\left([0;\overline{3,5}]^-\right) &= \frac{13}{11\sqrt{165}} = \frac{1}{10.8690\ldots},\\
\rho\left([0;\overline{5,6}]^-\right) &= \frac{589}{312\sqrt{195}} = \frac{1}{7.3970\ldots},\\
\rho\left([0;\overline{7,8}]^-\right) &= \frac{3649}{1664\sqrt{182}} = \frac{1}{6.1519\ldots},
\end{aligned}$$

from [10] give upper bounds on the optimal $C(R)$.

4. Proof of Theorem 3.1

We shall make frequent use of the following simple observation.

Lemma 4.1. *If $\lambda > \mu > 0$ then $f(z) = \dfrac{1-\lambda z}{1-\mu z}$ is decreasing for $0 \leq \lambda z < 1$.*

In particular, if $\lambda_1, \lambda_2 \geq 1$ and $0 \leq x \leq \alpha$, $0 \leq y \leq \beta$, with $\lambda_1\alpha, \lambda_2\beta < 1$, then

$$\frac{(1-\lambda_1 x)(1-\lambda_2 y)}{1-xy} \geq \left(\frac{1-\lambda_1\alpha}{1-\alpha y}\right)(1-\lambda_2 y) \geq \frac{(1-\lambda_1\alpha)(1-\lambda_2\beta)}{1-\alpha\beta}.$$

Proof. Plainly $f'(z) = -(\lambda-\mu)/(1-z\mu)^2 < 0$ for $0 \leq z < \mu^{-1}$. $\square$

Proof of Theorem 3.1. From (13) and (2) we have $s_2(k), s_4(k) \geq \frac{1}{4} \geq M(\alpha, \gamma^*)$, and Lemma 2.1 becomes

$$M(\alpha, \gamma^*) = \liminf_{k\to\infty} \min\{s_1(k), s_3(k)\}. \tag{25}$$

Since we are evaluating lim inf on k, from now on whenever we see the index k, it will be understood that we are letting $k \to \infty$. Also, we may assume that $a_i \geq R$ for all i.

Observe that changing the signs of t_i only interchanges $s_1(i)$ with $s_3(i)$. Hence, as long as we check both signs on the t_i, it will be enough to show that

$$s_3(k) \geq C(R).$$

We also observe that interchanging the pairs (a_{k-i}, t_{k-i}) with (a_{k+1+i}, t_{k+1+i}) for all $i \geq 0$ only interchanges $\bar{\alpha}_k$ with α_k and d_k^- with d_k^+.

The proof when R is even is straightforward.

Case I: R is even. In this case we have $R_* = R$, $R_{**} = R + 1$, and (14) becomes

$$\beta = [0; \overline{R}]^- \text{ and } \delta = \frac{1}{R+1-\beta} = [0; R+1, \overline{R}]^-,$$

with $\beta = \delta + \delta\beta > \delta$.

If a_k is odd and $t_k = 1$, then $d_k^- \leq \bar{\alpha}_k$, $d_k^+ \leq 0$, and

$$s_3(k) \geq \frac{(1-2\bar{\alpha}_k)(1-\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \geq \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)},$$

where the last inequality follows from the Lemma 4.1, since $\bar{\alpha}_k \leq \delta, \alpha_k \leq \beta$ by property (6). As observed above this also covers the case a_{k+1} odd with $t_{k+1} = 1$.

If a_k is odd and $t_k = -1$ with a_{k+1} even (likewise a_{k+1} odd, $t_{k+1} = -1$ with a_k even) we have $d_k^- \leq 0$, $d_k^+ \leq \alpha_k\alpha_{k+1} \leq \alpha_k\beta$ and Lemma 4.1 with $\alpha_k, \bar{\alpha}_k \leq \beta$ gives

$$s_3(k) \geq \frac{(1-\bar{\alpha}_k)(1-(1+\beta)\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \geq \frac{(1-\beta)(1-\beta-\beta^2)}{4(1-\beta^2)} \geq \frac{(1-\beta)(1-2\delta)}{4(1-\delta\beta)}, \tag{26}$$

since $\delta < \beta$ and $\beta + \beta^2 < 2\delta$ (equivalently $R \geq 2 + 2\beta$).

This just leaves the case that a_k and a_{k+1} both are even. If $d_k^- \leq 0$ and $d_k^+ \geq 0$ (likewise $d_k^- \geq 0$ and $d_k^+ \leq 0$) then $d_k^+ \leq \alpha_k\alpha_{k+1} \leq \alpha_k\beta$ and again we have (26).

Case II: R is odd. In this case we have $\delta = \beta + \delta\beta > \beta$, where

$$\beta = [0; \overline{R+1}]^- \quad \text{and} \quad \delta = \frac{1}{R-\beta} = [0; R, \overline{R+1}]^-.$$

We first establish some lemmas. Assume in both that $R \geq 3$ is odd and $\gamma = \gamma^*$.

Lemma 4.2. *Suppose that $\theta < 1$. If a_{k+1} is odd and $t_{k+1} = 1$, then*

$$\frac{1 - \alpha_k - d_k^+}{1 - \theta\alpha_k} \geq \frac{1 - 2\delta}{1 - \theta\delta}. \tag{27}$$

Likewise, if a_k is odd and $t_k = 1$, then

$$\frac{1 - \bar{\alpha}_k - d_k^-}{1 - \theta\bar{\alpha}_k} \geq \frac{1 - 2\delta}{1 - \theta\delta}. \tag{28}$$

P r o o f. Notice that it suffices to show the inequality (27) when $k = 0$. That is

$$A := \frac{1 - \alpha - d_0^+}{1 - \theta\alpha} \geq \frac{1 - 2\delta}{1 - \theta\delta},$$

where $\alpha_0 = \alpha = [0; a_1, a_2, \cdots]^-$, and $d_0^+ = t_1\alpha + t_2\alpha\alpha_1 + \cdots$

If $\alpha \leq \delta$ (for example the case when the a_i, $i \geq 2$, are all even), then

$$A \geq \frac{1 - 2\alpha}{1 - \theta\alpha} \geq \frac{1 - 2\delta}{1 - \theta\delta}, \tag{29}$$

from (13) and Lemma 4.1.

So suppose that $\alpha > \delta$ and let a_{n+1}, $n \geq 1$, be the odd partial quotient such that a_i is even for all $1 < i < n + 1$. Notice we must have $a_1 = a_{n+1} = R$ and $a_i = R + 1$ for $1 < i < n + 1$, else $\alpha < \delta$. Since $t_1 = 1$ and $t_{n+1} = -1$,

$$d_0^+ \leq \alpha - \alpha\alpha_1 \cdots \alpha_n + \alpha\alpha_1 \cdots \alpha_n\alpha_{n+1} \leq \alpha - \frac{1}{2}\alpha\alpha_1 \cdots \alpha_n,$$

and

$$A \geq \frac{1 - 2\alpha + \frac{1}{2}\alpha\alpha_1 \cdots \alpha_n}{1 - \theta\alpha} > \frac{1 - 2\alpha + \frac{1}{2}\alpha\alpha_1 \cdots \alpha_n}{1 - \theta\delta}. \tag{30}$$

Setting $\nu := [0; a_1, a_2, \ldots, a_n, a_{n+1} + 2, a_{n+2}, \ldots]^-$ we have $\nu < \delta$, and we just need to show that

$$\alpha - \nu \leq \frac{1}{4}\alpha\alpha_1 \cdots \alpha_n, \tag{31}$$

to obtain $1 - 2\alpha + \frac{1}{2}\alpha\alpha_1 \cdots \alpha_n \geq 1 - 2\nu \geq 1 - 2\delta$ and (27).

Recall that

$$\alpha = \frac{p_{n+1} - p_n\alpha_{n+1}}{q_{n+1} - q_n\alpha_{n+1}} = \frac{p_n - p_{n-1}\alpha_n}{q_n - q_{n-1}\alpha_n}. \tag{32}$$

Similarly,

$$\nu = \frac{p_{n+1} - p_n\alpha_{n+1} + 2p_n}{q_{n+1} - q_n\alpha_{n+1} + 2q_n},$$

and $p_{n+1}q_n - p_nq_{n+1} = 1$ gives

$$\alpha - \nu = \frac{2}{(q_{n+1} - q_n\alpha_{n+1})(q_{n+1} - q_n\alpha_{n+1} + 2q_n)} = \frac{2\alpha\alpha_1\cdots\alpha_n}{(q_{n+1} + (2-\alpha_{n+1})q_n)},$$

and (31) just needs $q_{n+1}+(2-\alpha_{n+1})q_n \geq 8$. Plainly $q_{n+1} \geq q_2 \geq 3\cdot 3-1 = 8$. □

Lemma 4.3. *Suppose that $\theta < 1$. If a_{k+1} is even and $d_k^+ \leq 0$, then*

$$\frac{1-\alpha_k - d_k^+}{1-\theta\alpha_k} \geq \frac{1-\beta}{1-\theta\beta}.$$

P r o o f. We proceed as in the proof of Lemma 4.2. Suppose $k=0$. Then we show

$$A := \frac{1-\alpha-d_0^+}{1-\theta\alpha} \geq \frac{1-\beta}{1-\theta\beta}.$$

If $\alpha \leq \beta$ then

$$A \geq \frac{1-\alpha}{1-\theta\alpha} \geq \frac{1-\beta}{1-\theta\beta}. \tag{33}$$

Assume $\alpha > \beta$, and let $a_{n+1}, n \geq 1$, be the odd partial quotient such that a_i is even for all $1 \leq i \leq n$. Then, since $t_1, t_2, \ldots, t_n = 0$ and $t_{n+1} = -1$, and trivially $\alpha_{n+1} \leq [0;\overline{3}]^- < 1/2$,

$$d_0^+ \leq -\alpha\alpha_1\cdots\alpha_n + \alpha\alpha_1\cdots\alpha_n\alpha_{n+1} \leq -\frac{1}{2}\alpha\alpha_1\cdots\alpha_n,$$

and

$$A \geq \frac{1-\alpha+\frac{1}{2}\alpha\alpha_1\cdots\alpha_n}{1-\theta\beta}.$$

Set $\nu := [0;a_1,a_2,\ldots,a_n,a_{n+1}+2,a_{n+2},\ldots]^- < \beta$. This time we just need to show $\alpha - \nu \leq \frac{1}{2}\alpha\alpha_1\cdots\alpha_n$, which reduces to $q_{n+1} + (2-\alpha_{n+1})q_n \geq 4$. Plainly, $q_{n+1} \geq q_2 \geq 3\cdot 4 - 1 = 11$. □

P r o o f o f T h e o r e m 3.1 w h e n R i s o d d.
We set $\sigma := [0;\overline{R}]^-$. We need to show that $s_3(k) \geq C(R)$. If a_k and a_{k+1} both are odd, then without loss of generality we can assume $t_k = -1$ and $t_{k+1} = 1$. Plainly $d_k^- \leq -\bar{\alpha}_k + \bar{\alpha}_k\bar{\alpha}_{k-1} \leq -\bar{\alpha}_k + \bar{\alpha}_k\sigma$, and by Lemma 4.2 and Lemma 4.1 (using $\sigma > \delta$ and $\bar{\alpha}_k \leq \sigma$) and $\sigma > \beta$

$$s_3(k) \geq \frac{(1-2\delta)(1-\bar{\alpha}_k\sigma)}{4(1-\bar{\alpha}_k\delta)} \geq \frac{(1-2\delta)(1-\sigma^2)}{4(1-\sigma\delta)} > \frac{(1-2\delta)(1-\sigma^2)}{4(1-\beta\delta)} > C(R),$$

since $\sigma^2 < \frac{1}{2}\sigma < \beta$.

Now it suffices to consider the following three cases: (i) a_k and a_{k+1} both are even, (ii) $(a_k, t_k) = (\text{odd}, -1)$ and a_{k+1} is even, (iii) $(a_k, t_k) = (\text{odd}, 1)$ and a_{k+1} is even. For (i) and (ii), it can be readily seen that

$$s_3(k) \geq \frac{1}{4}(1-\sigma)(1-\sigma-\sigma^2) = \frac{1}{4}(1-2\sigma+\sigma^3) \geq \frac{1}{4}(1-2\delta) \geq C(R),$$

using that

$$\sigma - \delta = (\sigma-\beta)\sigma\delta = (1-\beta+\sigma)\sigma^2\delta\beta < \frac{3}{2}\beta\sigma^3 < \frac{1}{2}\sigma^3.$$

For (iii), we apply Lemmas 4.2 and 4.3

$$\begin{aligned} s_3(k) &= \frac{(1-\bar{\alpha}_k - d_k^-)(1-\alpha_k-d_k^+)}{4(1-\bar{\alpha}_k\alpha_k)} \\ &\geq \frac{(1-2\delta)(1-\alpha_k-d_k^+)}{4(1-\delta\alpha_k)} \geq \frac{(1-2\delta)(1-\beta)}{4(1-\delta\beta)}. \end{aligned}$$

□

It remains just to demonstrate the asymptotics (19) and (20). For $R \geq 4$ even, one can use $R\beta = 1+\beta^2$ to write

$$E_1(R) = \frac{11-2\beta\,(6-3\beta+\beta^2)}{1\,+\,(1-2\beta)/R} = 11 + O\left(\frac{1}{R}\right),$$

with $E_1(4) = (524-256\sqrt{3})/11 = 7.3268\ldots$, $E_1(R) \nearrow 11$, and (19) is clear.

For $R \geq 3$ odd, one can use $R\beta = 1-\beta+\beta^2$ to write

$$E_2(R) = \frac{10-2\beta(11-7\beta+2\beta^2)}{1-2\beta/R} = 10 + O\left(\frac{1}{R}\right),$$

with $E_2(3) = (348-162\sqrt{3})/11 = 6.1279\ldots$, $E_2(R) \nearrow 10$, and (20) is clear. □

5. Proof of Theorem 3.2

We assume that α has expansion (5) of period (22) with $R \geq 4$ even.

Suppose first that γ has an expansion (8) with $t_i = 0$ when $a_i = R$ and $t_i = \pm 1$ when $a_i = R+1$, for all sufficiently large i. If $a_k = R+1$ and $t_k = 1$ then

$$\bar{\alpha}_k \to \delta,\ \alpha_k \to \beta, \quad d_k^- \to \delta,\ d_k^+ \to 0, \quad s_3(k) \to C(R), \quad \text{as } k, l \to \infty.$$

Likewise, if $a_k = R+1$ and $t_k = -1$, then $s_1(k) \to C(R)$ as $k, l \to \infty$. Hence these γ cannot contribute a value $M(\alpha,\gamma)$ strictly greater than $C(R)$ to $\rho(\alpha)$ as $l \to \infty$. By Theorem 3.1 we have $M(\alpha,\gamma^*) \geq C(R)$ and hence

$$\lim_{l\to\infty} M(\alpha,\gamma^*) = C(R). \tag{34}$$

It remains to show that $M(\alpha,\gamma) \le C(R)$ as $l \to \infty$ for the remaining $\gamma \notin \{m + l\alpha : m, l \in \mathbb{Z}\}$; that is, those γ having an expansion (8) with $|t_i| \ge 2$ infinitely often.

Observe that changing the signs of t_i only interchanges $s_1(i)$ with $s_3(i)$ and $s_2(i)$ with $s_4(i)$. Thus, if we eliminate any block of t_i from consideration, then the same will be true for its negative. Also, interchanging $\bar{\alpha}_k$ with α_k and d_k^- with d_k^+ does not change $s_1(k)$ and $s_3(k)$ (and interchanges $s_2(k), s_4(k)$). Hence, if we eliminate a block of t_i from consideration, then the same will be true for the reversed block of t_i (on a reversed block of a_i).

If $t_k = a_k$ infinitely often, then from [8, Lemma 1]

$$M(\alpha,\gamma) \le \liminf_{\substack{k\to\infty \\ t_k=a_k}} \frac{\bar{\alpha}_k}{4(1-\bar{\alpha}_k\alpha_k)}, \tag{35}$$

and hence

$$M(\alpha,\gamma) \le \frac{\beta}{4(1-\beta^2)} \le \frac{(\beta+\beta^2-\delta\beta)}{4(1-\beta^2+\beta^2-\delta\beta)} < C(R),$$

on observing that $(1-2\delta)(1-\beta) = 1-3\beta+4\delta\beta$ and $4\beta+\beta^2-5\delta\beta < 1$ (equivalently $R > 4-\delta(3-2\beta)$).

Thus, we may assume that $t_i = a_i$ at most finitely many times and the $|d_k^-| \le 1-\bar{\alpha}_k$ and $|d_k^+| \le 1-\alpha_k$ for suitably large k. Hence the terms in the $s_i(k)$ are all positive, and we notice that

$$\sqrt{s_3(k)s_4(k)} = \frac{\sqrt{((1-d_k^-)^2-\bar{\alpha}_k^2)(1-(\alpha_k+d_k^+)^2)}}{4(1-\bar{\alpha}_k\alpha_k)},$$

giving

$$\min\{s_3(k), s_4(k)\} \le \frac{1-d_k^-}{4(1-\bar{\alpha}_k\alpha_k)}, \tag{36}$$

since plainly

$$0 \le (1-d_k^-)^2-\bar{\alpha}_k^2 < (1-d_k^-)^2 \quad \text{and} \quad 0 \le 1-(\alpha_k+d_k^+)^2 < 1.$$

We establish the following lemmas. In each case we are assuming that $t_i = a_i$ at most finitely many times.

Lemma 5.1. *If the sequence $\{t_i\}_i$ in the expansion (8) of γ has infinitely many blocks of the form $t_k, t_{k+j} > 0$, for some $j > 0$, with at least one of a_k, a_{k+j} even, and $t_i = 0$ for any $k < i < k+j$, then $M(\alpha,\gamma) < C(R)$ as $l \to \infty$.*

Proof. Without loss of generality suppose that a_k is even. Then $a_k = R$, $t_k \geq 2$, and with $\theta = [0; \overline{R+1}]^-$ we have

$$\bar{\alpha}_k \geq \frac{1}{R-\theta} = \frac{\theta}{1-\theta}, \qquad \alpha_k \geq \delta \text{ as } l \to \infty.$$

Plainly $d_k^+ \geq 0$, while $d_k^- \geq 2\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- > \bar{\alpha}_k$ by (9), and we get

$$\begin{aligned} s_3(k) &\leq \frac{(1-2\bar{\alpha}_k)(1-\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \leq \frac{(1-\frac{2\theta}{1-\theta})(1-\delta)}{4(1-\delta\beta)} \\ &\leq \frac{1-3\theta}{4(1-\delta\beta)} < C(R), \end{aligned}$$

the second inequality from Lemma 4.1 and $\theta/(1-\theta) < \beta$, the third from $\delta > \theta$, and the last since $1-3\theta = 1-3\delta+3\delta\theta(\beta-\theta)$ while $(1-2\delta)(1-\beta) = 1-3\delta+\delta\beta$. The result follows from Lemma 2.1. □

We can now assume that the sequence $\{t_i\}_i$ in the expansion (8) eventually does not contain any block (or its negative) of the type excluded by Lemma 5.1.

Lemma 5.2. *If γ has infinitely many $t_k \geq 3$, then $M(\alpha,\gamma) < C(R)$.*

Proof. If $a_k = R$ and $t_k \geq 4$, then $d_k^- \geq 4\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- \geq 3\bar{\alpha}_k$ and by (36)

$$\begin{aligned} M(\alpha,\gamma) &\leq \frac{1-3\bar{\alpha}_k}{4(1-\bar{\alpha}_k\beta)} \leq \frac{1-3\delta}{4(1-\delta\beta)} \\ &= \frac{1-2\delta-\beta+\delta\beta}{4(1-\delta\beta)} < C(R), \end{aligned}$$

and hence we can assume that $|t_i| \leq 2$ if $a_i = R$. Suppose $a_k = R+1$ and $t_k \geq 3$. Then, $d_k^- \geq 3\bar{\alpha}_k - 2\bar{\alpha}_k\bar{\alpha}_{k-1}$, $d_k^+ \geq -2\alpha_k$, and as $l \to \infty$

$$\begin{aligned} s_3(k) &\leq \frac{(1-4\bar{\alpha}_k+2\bar{\alpha}_k\bar{\alpha}_{k-1})(1+\alpha_k)}{4(1-\bar{\alpha}_k\alpha_k)} \to \frac{(1-4\delta+2\delta\beta)(1+\beta)}{4(1-\delta\beta)} \\ &= \frac{(1-4\delta+\beta-2\delta\beta+2\delta\beta^2)}{4(1-\delta\beta)} \\ &= \frac{(1-2\delta-\beta+2\delta\beta^2)}{4(1-\delta\beta)} < C(R). \end{aligned}$$

So $M(\alpha,\gamma) < C(R)$ from Lemma 2.1 if this happens for infinitely many k. □

We now also assume that $|t_i| \le 2$ for all sufficiently large i.

Lemma 5.3. *If γ has infinitely many of the following blocks, then*
$$M(\alpha,\gamma) < C(R).$$
(i) $t_k = 1$ *and* $t_{k+1} = -2$, *or*

(ii) $t_k = 0$ *and* $t_{k+1} = -2$.

Proof.
(i) Plainly $d_k^- \le \bar{\alpha}_k$, $d_k^+ \le -2\alpha_k + 2\alpha_k\alpha_{k+1}$, and as $l \to \infty$
$$\begin{aligned} s_1(k) &\le \frac{(1-3\alpha_k+2\alpha_k\alpha_{k+1})}{4(1-\bar{\alpha}_k\alpha_k)} \to \frac{(1-3\beta+2\beta^2)}{4(1-\delta\beta)} \\ &= \frac{(1-2\delta-\beta+2\delta\beta^2)}{4(1-\delta\beta)} < C(R). \end{aligned}$$

(ii) With $\theta = [0;\overline{R+1}]^-$ we have $\alpha_k \ge \dfrac{1}{R-\theta} =: \lambda$. Since
$$d_k^- \le 2\bar{\alpha}_k\bar{\alpha}_{k-1} \le 2\beta\bar{\alpha}_k, \qquad d_k^+ \le -2\alpha_k + 2\alpha_k\alpha_{k+1} \quad \text{and} \quad \alpha_k \le \beta$$
$$\begin{aligned} s_1(k) &\le \frac{(1-(1-2\beta)\bar{\alpha}_k)(1-3\alpha_k+2\beta^2)}{4(1-\bar{\alpha}_k\beta)} \\ &\le \frac{(1-\delta+2\beta\delta)(1-3\lambda+2\beta^2)}{4(1-\delta\beta)} \\ &\le \frac{(1-3\lambda+2\beta^2)}{4(1-\delta\beta)} < C(R), \end{aligned}$$
using Lemma 4.1 and $\bar{\alpha}_k > \delta$ for the second inequality. For the last inequality observe that
$$\beta - \lambda = \beta\lambda(\beta-\theta) < \lambda\beta^2$$
so that
$$1-3\lambda+2\beta^2 < 1-3\beta+\beta^2(2+3\lambda) \quad \text{while} \quad (1-2\delta)(1-\beta) = 1-3\beta+\beta^2(4-4\delta).$$
Infinitely many such k would give $M(\alpha,\gamma) < C(R)$ by Lemma 2.1. $\square$

From Lemmas 5.2 and 5.3 we see that a γ with infinitely many $|t_i| \ge 2$ has $M(\alpha,\gamma) \le C_0(R) < C(R)$ as $l \to \infty$ (where $C_0(R)$ is made explicit in the proof). Hence
$$\lim_{l\to\infty} \rho(\alpha) = \lim_{l\to\infty} M(\alpha,\gamma^*) = C(R).$$

6. Proof of Theorem 3.3

Suppose that

$$\alpha=[0;\overline{R,R,R+1,R,R+1,R+1,R,R+1,R+1,R,R+1}]^{-} \quad \text{with } R \text{ odd.}$$

By Theorem 3.1 and (20) we have

$$\rho(\alpha)\geq M(\alpha,\gamma^{*})\geq C(R)=\frac{1}{4}\left(1-\frac{3}{R}+\frac{4}{R^{2}}+O(R^{-3})\right),$$

so we just need to show that all γ have

$$M(\alpha,\gamma)\leq\frac{1}{4}\left(1-\frac{3}{R}+\frac{4}{R^{2}}+O(R^{-3})\right). \tag{37}$$

We observe the following

$$[0;R,R \text{ or } R+1,\cdots]^{-}=\frac{1}{R}+O(R^{-3}),$$
$$[0;R+1,R \text{ or } R+1,\cdots]^{-}=\frac{1}{R}-\frac{1}{R^{2}}+O(R^{-3}),$$

and so for α we have

$$\frac{1}{1-\bar{\alpha}_k\alpha_k}=1+\bar{\alpha}_k\alpha_k+(\bar{\alpha}_k\alpha_k)^{2}+\cdots=1+\frac{1}{R^{2}}+O(R^{-3}). \tag{38}$$

Now if γ has $t_k=a_k$ infinitely often, then from (35)

$$M(\alpha,\gamma)\leq\frac{1}{4}\left(\frac{1}{R}+O(R^{-3})\right)\left(1+\frac{1}{R^{2}}+O(R^{-3})\right)=\frac{1}{4}\left(\frac{1}{R}+O(R^{-3})\right),$$

so we can assume that γ has only finitely many $t_i=a_i$. In view of (38) we write

$$\tilde{s}_j(k)=4(1-\bar{\alpha}_k\alpha_k)s_j(k), \quad j=1,\ldots,4,$$

and (37) amounts to showing that there are infinitely many k with an

$$\tilde{s}_j(k)\leq 1-\frac{3}{R}+\frac{3}{R^{2}}+O(R^{-3}). \tag{39}$$

We proceed as in the proof of Theorem 3.2 successively eliminating blocks of t_i, recalling that when we eliminate a block we also eliminate its negative or reverse (by interchanging $s_j(k)$).

By (36) we will get (39) if γ has infinitely many k with

$$d_k^{-}\geq\frac{3}{R}-\frac{3}{R^{2}}+O(R^{-3}). \tag{40}$$

We use this to rule out large $|t_i|$. If $t_k\geq 5$ then we have

$$d_k^{-}\geq 5\bar{\alpha}_k+\bar{\alpha}_k d_{k-1}^{-}>4\bar{\alpha}_k\geq\frac{4}{R}+O(R^{-2}).$$

If we have $t_k = 4$ with $|t_{k-1}| \leq 4$ then $d_{k-1}^- = O(R^{-1})$ and again

$$d_k^- = \frac{4}{R} + O(R^{-2}).$$

So we can assume that $|t_i| \leq 3$ for all but finitely many i. If we have infinitely many blocks $a_k, a_{k-1} = R, R+1$ (or their reverse) with $t_k = 3$, then

$$d_k^- \geq 3\bar{\alpha}_k - 2\bar{\alpha}_k\bar{\alpha}_{k-1} + O(R^{-3}) = \frac{3}{R} - \frac{2}{R^2} + O(R^{-3}).$$

Hence we can assume that (all but finitely many) $t_i = \pm 1$ if $a_i = R$ and $t_i = 0, \pm 2$ if $a_i = R+1$.

First we rule out infinitely many consecutive positive or consecutive negative t_i. If $t_k, t_{k+1} > 0$ then $d_k^- \geq \bar{\alpha}_k + O(R^{-2}), d_k^+ \geq \alpha_k + O(R^{-2})$ and

$$\tilde{s}_3(k) \leq \big(1 - 2\bar{\alpha}_k + O(R^{-2})\big)\big(1 - 2\alpha_k + O(R^{-2})\big) = 1 - \frac{4}{R} + O(R^{-2}).$$

Next we rule out infinitely many blocks $t_{k-1}, t_k, t_{k+1}, t_{k+2} = 0, 1, 0, 0$ (or their reverse $0, 0, 1, 0$ or their negatives) since $d_k^- = \bar{\alpha}_k + O(R^{-3})$, $d_k^+ = O(R^{-3})$ and

$$\begin{aligned}
\tilde{s}_3(k) &= \big(1 - 2\bar{\alpha}_k + O(R^{-3})\big)\,\big(1 - \alpha_k + O(R^{-3})\big) \\
&= \left(1 - \frac{2}{R} + O(R^{-3})\right)\left(1 - \frac{1}{R} + \frac{1}{R^2} + O(R^{-3})\right) \\
&= 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}).
\end{aligned}$$

If $t_k, t_{k+1} = 2, 0$, then $d_k^- = 2\bar{\alpha}_k + O(R^{-2})$, $d_k^+ = O(R^{-2})$ and

$$\begin{aligned}
\tilde{s}_3(k) &= \big(1 - 3\bar{\alpha}_k + O(R^{-2})\big)\,\big(1 - \alpha_k + O(R^{-2})\big) \\
&= \left(1 - \frac{3}{R} + O(R^{-2})\right)\left(1 - \frac{1}{R} + O(R^{-2})\right) \\
&= 1 - \frac{4}{R} + O(R^{-2}).
\end{aligned}$$

Hence blocks $R+1, R+1$ must eventually have $t_k, t_{k+1} = 0,0$ or $2,-2$ or $-2,2$. If $t_{k-1}, t_k, t_{k+1} = -2,1,-2$, then $d_k^- = \bar{\alpha}_k - 2\bar{\alpha}_k\bar{\alpha}_{k-1} + O(R^{-3})$, $\quad d_k^+ \leq -2\alpha_k + 2\alpha_k\alpha_{k+1} + O(R^{-3})$ and

$$\begin{aligned}\tilde{s}_1(k) &\leq \left(1 - 2\bar{\alpha}_k\bar{\alpha}_{k-1} + O(R^{-3})\right)\left(1 - 3\alpha_k + 2\alpha_k\alpha_{k+1} + O(R^{-3})\right)\\ &= \left(1 - \frac{2}{R^2} + O(R^{-3})\right)\left(1 - \frac{3}{R} + \frac{5}{R^3} + O(R^{-3})\right)\\ &= 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}).\end{aligned}$$

Hence if we have a block $a_{k-2},\ a_{k-1},\ a_k,\ a_{k+1},\ a_{k+2} = R+1,\ R+1,\ R,\ R+1,\ R+1$ with $t_k = \pm 1$ then we must have $t_{k+1}, t_{k+2} = 0,0$ and $t_{k-1}, t_{k-2} = \mp 2, \pm 2$ (or vice versa in which case we use the reverse). Consider then the block

$$a_{k+1}, \ldots, a_{k+6} = R+1,\ R+1,\ R, R+1,\ R,\ R, \quad t_{k+1},\ t_{k+2} = 0,0.$$

Assuming that $t_{k+3} = 1$ (or use the negative), then having ruled out $0,0,1,0$, we must have $t_{k+4}, t_{k+5}, t_{k+6} = -2,1,-1$ and finally

$$\begin{aligned}\tilde{s}_1(k+4) &= \left(1 - 3\bar{\alpha}_{k+4} + \bar{\alpha}_{k+4}\bar{\alpha}_{k+3} + O(R^{-3})\right)\left(1 - \alpha_{k+4}\alpha_{k+5} + O(R^{-3})\right)\\ &= \left(1 - \frac{3}{R} + \frac{4}{R^2} + O(R^{-3})\right)\left(1 - \frac{1}{R^2} + O(R^{-3})\right)\\ &= 1 - \frac{3}{R} + \frac{3}{R^2} + O(R^{-3}).\end{aligned}$$

□

7. Proof of (24)

If in the previous proof we had taken α to have period (23) with $R \geq 5$ odd, then (37) would still hold, except for those γ whose t_i eventually consist of zeros one side of the ± 1 and blocks of $\mp 2, \pm 2$ the other. For these γ, if $t_k = 1$ inside a block $\ldots, 0, 0, 1, -2, 2, \ldots$, then

$$d_k^- \to \delta, \quad d_k^+ \to -2\beta/(1+\beta) \quad \text{and} \quad d_{k-1}^- \to 0, \quad d_{k-1}^+ \to \delta - 2\delta\beta/(1+\beta) \ \text{ as } l \to \infty$$

and

$$s_3(k-1),\ s_1(k) \to \frac{1 - 3\beta + \frac{2\beta^2}{1+\beta}}{4(1-\delta\beta)} = \frac{\left(1 - 2\delta + \frac{2\delta\beta}{1+\beta}\right)(1-\beta)}{4(1-\delta\beta)} =: C_1(R),$$

with

$$s_1(k-1) > s_3(k-1) \quad \text{and} \quad s_3(k) > (1-2\delta)/4(1-\alpha\beta) > C_1(R).$$

Likewise for the negatives and reverses. At all places

$$s_2(k), s_4(k) > (1-\bar{\alpha}_k)(1-\alpha_k)/4(1-\beta\alpha_k) > (1-\delta)^2/4(1-\delta\beta) > C_1(R).$$

If $t_k, t_{k+1} = 0, 0$ and $d_k^- \to 0$, $\bar{\alpha}_k \to \beta$ (likewise if $d_k^+ \to 0$, $\alpha_k \to \beta$), then

$$\begin{aligned} s_1(k),\ s_3(k) &\geq \frac{(1-\beta)(1-\alpha_k(1+\delta))}{4(1-\beta\alpha_k)} \\ &> \frac{(1-\beta)(1-\delta-\delta^2)}{4(1-\delta\beta)} > C_1(R). \end{aligned}$$

If $t_k, t_{k+1} = -2, 2$, then $d_{k-1} > \beta$ (either $t_{k-1} = 2$ or $t_{k-1} = 1$ with $d_{k-1} \to \delta$) and

$$\begin{aligned} s_1(k) &\geq \frac{(1-\bar{\alpha}_k(3-\beta))\,(1+\beta(1-2\delta))}{4(1-\bar{\alpha}_k\beta)} \\ &\geq \frac{(1-3\delta+\delta\beta)(1+\beta-2\beta\delta)}{4(1-\delta\beta)} > C_1(R), \end{aligned}$$

for $R \geq 9$ using

$$(1-3\delta+\delta\beta)(1+\beta-2\beta\delta) = (1-3\beta+2\beta^2) + \beta(1-9\beta+4\delta^2\beta),$$

replacing $\bar{\alpha}_k$ by $1/(R+1-\delta)$ instead of δ in the second inequality and checking numerically for $R = 5$ and 7. Likewise for $s_3(k)$ and for $2, -2$. Hence these γ have $M(\alpha, \gamma) \to C_1(R)$ as $l \to \infty$. The proof of Theorem 3.3 immediately gives (24) for suitably large R. To see that it is true for all $R \geq 5$ we show that $M(\alpha, \gamma) < C_1(R)$ for the other γ. Notice, if we let $l \to \infty$, then $(1-\bar{\alpha}_k\alpha_k)^{-1} \leq (1-\delta\beta)^{-1}$; so it will be enough to show that the remaining γ have infinitely many

$$\tilde{s}_j(k) \leq 1 - 3\beta + \frac{2\beta^2}{1+\beta} = \left(1 - 2\delta + \frac{2\delta\beta}{1+\beta}\right)(1-\beta) =: \tilde{C}_1(R).$$

We repeat the steps of the proof of Theorem 3.3; successive ruling out certain blocks of t_i (or their negatives and reverses) occurring infinitely often. We can rule out $t_k = a_k$ since $\delta < 1 - 3\beta < \tilde{C}_1(R)$. To eliminate large $|t_k|$ we replace (40) by

$$d_k^- \geq 3\beta - \frac{2\beta^2}{1+\beta},$$

successively ruling out infinitely many $t_k \geq 4$ using $d_k^- > 3\beta$, then $t_k = 3$ using

$$d_k^- \geq 3\delta - \frac{2\delta\beta}{1-\beta} > 3\beta.$$

Now if $t_k = 2$ and $t_{k+j} > 0$ for some $j \geq 1$, with $t_i = 0$ for any $k < i < k+j$, then

$$d_k^- = 2\bar{\alpha}_k + \bar{\alpha}_k d_{k-1}^- \geq \bar{\alpha}_k + \bar{\alpha}_k \bar{\alpha}_{k-1}, d_k^+ \geq 0,$$

and

$$\tilde{s}_3(k) \leq (1 - 2\beta - \beta^2)(1 - \beta) = (1 - 2\delta + 2\delta\beta - \beta^2)(1 - \beta) < \tilde{C}_1(R).$$

If $t_k, t_{k+1} = 2, 0$, then

$$\tilde{s}_3(k) \leq (1 - 3\beta + 2\beta\delta)(1 - \beta + 2\beta\delta) = 1 - 3\beta + 2\beta^2 - 2\beta^3 - \lambda < \tilde{C}_1(R),$$

with

$$\lambda = \beta\big(1 - 5\beta + 2\beta^2(1 - 2\delta^2)\big) > 0.$$

If $t_{k-1}, t_k, t_{k+1} = 0, 1, 0$, then as $l \to \infty$,

$$\tilde{s}_3(k) \to (1 - 2\delta)(1 - \beta) < \tilde{C}_1(R).$$

Finally, if $t_{k-1}, t_k, t_{k+1} = -2, 1, -2$, then

$$\tilde{s}_1(k) \leq \left(1 - \frac{2\delta\beta}{1+\beta}\right)\left(1 - 3\beta + \frac{2\beta^2}{1+\beta}\right) < \tilde{C}_1(R).$$

□

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