sciendo

uniform distribution theory

DOI: 10.2478/UDT-2023-0014
Unif. Distrib. Theory **18** (2023), no.2, 57–76

ON JORDAN DOUBLE SUMS AND RELATED SUMMATORY FUNCTIONS

Sanying Shi—Michel Weber

School of Mathematics and Physics, Queen's University Belfast, Belfast, UNITED KINGDOM

ABSTRACT. We study the Jordan double sums

$$\sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}},$$

which are connected with usual zeta approximating sums. Some other related summatory functions involving Jordan's function are considered and estimated in this paper.

Communicated by Florian Luca

1. Introduction

Consider the following double sums

$$\sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon+it}}, \tag{1.1}$$

in which $J_\varepsilon(d)$ is the generalized Jordan function, $J_\varepsilon(d) = \sum_{h|d} h^\varepsilon \mu(\frac{d}{h})$, μ being Möbius function.

2010 Mathematics Subject Classification: Primary: 11A25, Secondary: 11C99.
Keywords: generalized Jordan function, double Jordan sum, summatory function, Dirichlet polynomials, Riemann Zeta function.

These sums are naturally connected with zeta approximating sums, hence our interest in their study. Consider indeed two different zeta approximating sums

$$\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}},\qquad \text{and}\qquad \sum_{n=1}^{M}\frac{1}{n^{\sigma+\varepsilon+it}},$$

where $N\ge 1$, $M\ge 1$ and σ,ε,t are real numbers. By perturbating the first of a factor n^{ε} and using Möbius inversion formula, we get

$$\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}}=\sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d^{\sigma+\varepsilon+it}}\sum_{m=1}^{\lfloor\frac{N}{d}\rfloor}\frac{1}{m^{\sigma+\varepsilon+it}}. \tag{1.2}$$

Replacing ε by 2ε in (1.2) yields similarly,

$$\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}}=\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon+it}}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m^{\sigma+2\varepsilon+it}}. \tag{1.3}$$

For $\sigma=0$ this gives,

$$\sum_{n=1}^{N}\frac{1}{n^{it}}=\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon+it}}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m^{2\varepsilon+it}}.$$

The notation (d,δ), $[d,\delta]$, respectively, stands for the greatest common divisor and least common multiple of the numbers d and δ. This provides a way to connect a zeta approximating sum with another oflarger exponent, thus of a smoother behavior, and thereby, by approximation, to connect the Riemann zeta function $\zeta(\sigma+it)$ at two different values of σ, which is is a quite important question. It is expected from the study made in our paper that in a subsequent work, new results for zeta approximating sums and the Riemann zeta function can be derived.

Proposition 1.1.
(i) *For any $N\ge 1$, $k\ge 1$ and any real numbers σ,ε,t,*

$$\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}}=\sum_{[d_1,\ldots,d_k]\le N}\frac{J_\varepsilon(d_1)\ldots J_\varepsilon(d_k)}{[d_1,\ldots,d_k]^{\sigma+k\varepsilon+it}}\sum_{m=1}^{\frac{N}{[d_1,\ldots,d_k]}}\frac{1}{m^{\sigma+k\varepsilon+it}}.$$

(ii) *For any $N\ge 1$, and any real numbers σ,ε,t, we have*

$$\Big|\sum_{n=1}^{N}\frac{1}{n^{\sigma+it}}\Big|\le\Big(\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}\Big)\max_{M\in\mathcal{E}(N)}\Big|\sum_{m=1}^{M}\frac{1}{m^{\sigma+2\varepsilon+it}}\Big|,$$

where $\mathcal{E}(N)$ denotes the set of integers of the form $\lfloor\frac{N}{d}\rfloor$, d integer, $1\le d\le N$.

(iii) *Let $\zeta(\sigma+it)$ be the Riemann zeta-function. Let $\sigma_0>0$. Then for any $|t|\ge 1$, $\sigma\ge\sigma_0$,*

$$|\zeta(\sigma+it)| \le \left(\sum_{[d,\delta]\le|t|} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}\right) \max_{M\in\mathcal{E}(|t|)} \left|\sum_{m=1}^{M} \frac{1}{m^{\sigma+2\varepsilon+it}}\right|$$

$$+ C_{\sigma_0}|t|^{-\sigma},$$

where C_{σ_0} depends on σ_0 only.

P r o o f. Assertions (i) and (ii) are immediate. Now from the approximation formula for the Riemann zeta-function we have

$$\sup_{|t|\ge1,\sigma\ge\sigma_0} |t|^\sigma \left|\zeta(\sigma+it) - \sum_{k=1}^{|t|} \frac{1}{k^{\sigma+it}}\right| < \infty.$$

See [6, Th. 4.11]. The third inequality then follows from (ii). □

We study the double sums

$$\sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}},$$

as well as the related summatory functions

$$\text{(I)} \qquad \sum_{1\le d\le x} \frac{J_\varepsilon(d)}{d^\varepsilon},$$

$$\text{(II)} \qquad \sum_{n\le x} \left(\frac{J_\varepsilon(n)}{n^\varepsilon}\right)^2,$$

$$\text{(III)} \qquad \sum_{n=1}^{N} \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}}, \qquad 0<\theta<1\,.$$

We obtain in the following sections sharp estimates of these sums.

2. Estimates of the double sum $\sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}$

For some range of values of σ, ε at least, these sums can be estimated directly starting from (1.2) and (1.3).

Theorem 2.1.
(i) *Assume that* $0<\varepsilon<1/2$. *Then,*

$$c_1(\varepsilon)\,N^{2\varepsilon}\le\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}\ \le\ c_2(\varepsilon)\,N^{2\varepsilon}.$$

Also,

$$c_3(\varepsilon)N^{\varepsilon}\le\sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d}\ \le\ c_4(\varepsilon)N^{\varepsilon}\ .$$

If $\varepsilon>1/2$, *then*

$$N\le\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}\ \le\ c_5(\varepsilon)N^{2\varepsilon}$$

(ii) *If* $0<\sigma<1$, *then*

$$\sum_{[d,\delta]\le N}\frac{1}{[d,\delta]^{\sigma}}=\frac{3}{\pi^2}\cdot\frac{1}{1-\sigma}N^{1-\sigma}(\log N)^2 + \mathcal{O}\left(N^{1-\sigma}(\log N)\right),$$

and

$$c_6(\sigma)\,e^{-c(\varepsilon)\frac{(\log N)^{1-\varepsilon}}{\log\log N}}N^{1-\sigma}(\log N)^2\le\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}} \le c_7(\sigma)\,N^{1-\sigma}(\log N)^2.$$

Further, if $\sigma=1$,

$$\sum_{[d,\delta]\le N}\frac{1}{[d,\delta]}=\frac{1}{\pi^2}(\log N)^3+\mathcal{O}\left((\log N)^2\right),$$

and

$$c_8(\varepsilon)\,e^{-c(\varepsilon)\frac{(\log N)^{1-\varepsilon}}{\log\log N}}(\log N)^3\le\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{1+2\varepsilon}}\le c_9(\varepsilon)(\log N)^3.$$

(iii) *If* $\sigma+2\varepsilon>1$, *then*

$$c_{10}(\sigma)\,e^{-c(\varepsilon)\frac{(\log N)^{1-\varepsilon}}{\log\log N}}N^{1-\sigma}\le\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}} \le c_{11}(\sigma)\,N^{1-\sigma}.$$

Here $c_i(\varepsilon)$ *(resp.* $c_j(\sigma)$*) are positive constants depending on* ε *(resp.* σ*) only.*

Before doing the proof, we notice that

$$\sum_{n=1}^{N}\frac{1}{n^{\sigma}} = \sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d^{\sigma+\varepsilon}}\sum_{m=1}^{\frac{N}{d}}\frac{1}{m^{\sigma+\varepsilon}} = \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m^{\sigma+2\varepsilon}}, \tag{2.1}$$

which follows from (1.2), (1.3), letting $t=0$.

P r o o f. Using (2.1) with $\sigma = 1-2\varepsilon$ yields, for $\varepsilon \ge 0$,

$$C(\varepsilon)N^{2\varepsilon} \ge \sum_{n=1}^{N}\frac{1}{n^{1-2\varepsilon}} = \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m} \ge \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}.$$

(i) Assume that $\varepsilon < 1/2$. Using (2.1) with $\sigma = 0$ gives

$$N = \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon}}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m^{2\varepsilon}} \le C(\varepsilon)\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon}}\left(\frac{N}{[d,\delta]}\right)^{1-2\varepsilon} = C(\varepsilon)N^{1-2\varepsilon}\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}.$$

Thus

$$\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]} \ge C(\varepsilon)N^{2\varepsilon}.$$

So that

$$c_1(\varepsilon)N^{2\varepsilon} \le \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]} \le c_2(\varepsilon)N^{2\varepsilon}.$$

Now trivially

$$\sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d} \le C(\varepsilon)N^{\varepsilon}.$$

Next, by letting $\sigma = 1-2\varepsilon$ in (2.1),

$$C(\varepsilon)N^{\varepsilon}\sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d} \ge \sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d^{1-\varepsilon}}\sum_{m=1}^{\frac{N}{d}}\frac{1}{m^{1-\varepsilon}}$$

$$= \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m} \ \ge\ c(\varepsilon)N^{2\varepsilon}.$$

Consequently,

$$c_1(\varepsilon)N^{\varepsilon} \ \le \sum_{1\le d\le N}\frac{J_\varepsilon(d)}{d} \ \le\ c_2(\varepsilon)N^{\varepsilon}.$$

If $\varepsilon > 1/2$. Similarly,

$$N = \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon}}\sum_{m=1}^{\frac{N}{[d,\delta]}}\frac{1}{m^{2\varepsilon}}$$

$$\le \left(\sum_{m\ge 1}\frac{1}{m^{2\varepsilon}}\right)\sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon}}.$$

Whence

$$c(\varepsilon)N^{2\varepsilon} \ge \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]} \ge \sum_{[d,\delta]\le N}\frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{2\varepsilon}} \ge N.$$

(ii) Assume that $0 < \sigma \le 1$, and let $R(u) = \#\{(\delta, d) \in \mathbb{N}^2 : [d,\delta] = u\}$. Clearly, $R(u) \le d^2(u)$ (in fact $\sum_{v|u} R(v) = d^2(u)$, where d is the divisor function; which by Möbius inversion formula implies $R = \mu * d^2$, whence R is a multiplicative function). Consequently,

$$\sum_{[d,\delta]\le N}\frac{1}{[d,\delta]^{\sigma}} = \sum_{u\le N}\frac{1}{u^{\sigma}}\#\big\{(\delta,d)\in\mathbb{N}^2 : [d,\delta]=u\big\}$$

$$= \sum_{u\le N}\frac{R(u)}{u^{\sigma}} = \sum_{u\le N}\frac{1}{u^{\sigma}}\sum_{d|u}\mu(d)d^2\left(\frac{u}{d}\right)$$

$$= \sum_{d\le N}\frac{\mu(d)}{d^{\sigma}}\sum_{u\le N/d}\frac{d^2(u)}{u^{\sigma}}. \tag{2.2}$$

For $0 < \sigma < 1$, by using Abel summation, and the formula ([7]),

$$\sum_{n\le x} d^2(n) = \frac{1}{\pi^2}x(\log x)^3 + Bx(\log x)^2 + Cx\log x + \mathcal{O}(x),$$

where B, C are the constants.

We get

$$\sum_{u\le N}\frac{d^2(u)}{u^\sigma}=D_1N^{1-\sigma}(\log N)^3+\ D_2N^{1-\sigma}(\log N)^2 \\ +\ D_3N^{1-\sigma}\log N+\mathcal{O}(N^{1-\sigma})\,,$$

where

$$D_1=\frac{1}{(1-\sigma)}\cdot\frac{1}{\pi^2},\qquad D_2=B+\frac{\sigma B}{1-\sigma}-\frac{3\sigma}{\pi^2(1-\sigma)^2}$$

and

$$D_3=\frac{C}{1-\sigma}-\frac{2\sigma B}{(1-\sigma)^2}+\frac{6\sigma}{\pi^2(1-\sigma)^3}\,,$$

$$\begin{aligned}\sum_{[d,\delta]\le N}\frac{1}{[d,\delta]^\sigma}&=\sum_{d\le N}\frac{\mu(d)}{d^\sigma}\left(D_1\left(\frac{N}{d}\right)^{1-\sigma}\left(\log\left(\frac{N}{d}\right)\right)^3\right.\\&\quad+\ D_2\left(\frac{N}{d}\right)^{1-\sigma}\left(\log\left(\frac{N}{d}\right)\right)^2\\&\quad\left.+\ D_3\left(\frac{N}{d}\right)^{1-\sigma}\left(\log\left(\frac{N}{d}\right)\right)+\mathcal{O}\left(\left(\frac{N}{d}\right)^{1-\sigma}\right)\right)\\&=D_1N^{1-\sigma}\sum_{d\le N}\frac{\mu(d)}{d}\left(\log\left(\frac{N}{d}\right)\right)^3\\&\quad+\ D_2N^{1-\sigma}\sum_{d\le N}\frac{\mu(d)}{d}\left(\log\left(\frac{N}{d}\right)\right)^2\\&\quad+\ D_3N^{1-\sigma}\sum_{d\le N}\frac{\mu(d)}{d}\left(\log\left(\frac{N}{d}\right)\right)+\mathcal{O}\left(N^{1-\sigma}(\log N)\right)\\&=3D_1N^{1-\sigma}(\log N)^2+\mathcal{O}\left(N^{1-\sigma}(\log N)\right)\\&=\frac{3}{\pi^2}\cdot\frac{1}{1-\sigma}N^{1-\sigma}(\log N)^2+\mathcal{O}\left(N^{1-\sigma}(\log N)\right),\end{aligned}$$

where we have used the estimates for $k=2,3$ (see [5])

$$\sum_{n\le x}\frac{\mu(n)}{n}\left(\log\frac{x}{n}\right)^k=\ k(\log x)^{k-1}+\sum_{i=1}^{k-2}c_i^{(k)}\left((\log x)^i\right)+\mathcal{O}(1)\,,$$

and

$$\sum_{n\le x}\frac{\mu(n)}{n}\log\frac{x}{n}=\mathcal{O}(1)\,.$$

For $\sigma = 1$, a similar proof of the case $0 < \sigma < 1$ yields

$$\sum_{[d,\delta]\le N} \frac{1}{[d,\delta]} = \frac{1}{\pi^2}(\log N)^3 + \mathcal{O}\left((\log N)^2\right).$$

The remaining lower bound part is trivial. It is obvious now that for any $\varepsilon > 0$,

$$\sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{1+2\varepsilon}} \le (\log N)^3.$$

As to the claimed lower bound, it now follows from Lemma 3.1.

(iii) If $\sigma + 2\varepsilon > 1$, then

$$c(\sigma)N^{1-\sigma} = \sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}} \sum_{m=1}^{\frac{N}{[d,\delta]}} \frac{1}{m^{\sigma+2\varepsilon}} \le c(\sigma+2\varepsilon) \sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}.$$

Also we have

$$c(\sigma)N^{1-\sigma} = \sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}} \sum_{m=1}^{\frac{N}{[d,\delta]}} \frac{1}{m^{\sigma+2\varepsilon}} \ge \sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}}.$$

Then we obtain

$$c_2(\sigma)N^{1-\sigma} \le \sum_{[d,\delta]\le N} \frac{J_\varepsilon(d)J_\varepsilon(\delta)}{[d,\delta]^{\sigma+2\varepsilon}} \le c_3(\sigma)N^{1-\sigma}.$$ □

3. Estimates of $J_\varepsilon(n)$ and $\sum_{1\le d\le x} \frac{J_\varepsilon(d)}{d^\varepsilon}$

Recall before continuing that $J_\varepsilon(n)$ is a multiplicative function, $J_\varepsilon(1)=1$, $J_\varepsilon(p^\alpha) = p^{\alpha\varepsilon}(1-p^{-\varepsilon})$ for p prime, $\alpha \ge 1$. Further recall that $J_1 = \phi$, the Euler totient function and $J_0 = \delta$, the inverse Möbius function. We first collect individual estimates.

Lemma 3.1. *We have*

$$C\frac{v}{\log\log v} \le \phi(v) \le v.$$

If $0 < \varepsilon < 1$,

$$v^\varepsilon \exp\left\{-c(\varepsilon)\frac{(\log v)^{1-\varepsilon}}{\log\log v}\right\} \le J_\varepsilon(v) \le v^\varepsilon.$$

Further,

$$\sum_{1\le d\le x}\frac{J_\varepsilon(d)}{d^\varepsilon}=\frac{1}{\zeta(1+\varepsilon)}x+\mathcal{O}(x^{1-\varepsilon})\,,$$

and as $\varepsilon\to 0$,

$$\sum_{1\le d\le x}\frac{J_\varepsilon(d)}{d^\varepsilon}=x\sum_{u=1}^{\infty}\frac{\mu(u)}{u^{1+\varepsilon}}+\mathcal{O}(x)\ll\frac{x}{\varepsilon}\,.$$

P r o o f. The first estimate is well-known and is proved in [4, p. 55] using "record's breaking numbers". The second estimate can be proved using the same method. Now,

$$\frac{J_\varepsilon(d)}{d^\varepsilon}=\sum_{u|d}\frac{\mu(u)}{u^\varepsilon}=\prod_{p|d}(1-p^{-\varepsilon})\,.$$

Further,

$$\sum_{u\le x}\frac{\mu(u)}{u^{1+\varepsilon}}=\frac{1}{\zeta(1+\varepsilon)}+\mathcal{O}(x^{-\varepsilon})\,. \tag{3.1}$$

Hence by [4, (2.3), p. 36],

$$\sum_{1\le d\le x}\frac{J_\varepsilon(d)}{d^\varepsilon}=x\sum_{u=1}^{\infty}\frac{\mu(u)}{u^{1+\varepsilon}}+\mathcal{O}(x^{1-\varepsilon})=\frac{1}{\zeta(1+\varepsilon)}\,x+\mathcal{O}(x^{1-\varepsilon})\,.$$

Letting $x\to\infty$ in (3.1),

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1+\varepsilon}}\ll\frac{1}{\varepsilon}+1\,. \tag{3.2}$$

Then for $\varepsilon\to 0$,

$$\sum_{1\le d\le x}\frac{J_\varepsilon(d)}{d^\varepsilon}=x\sum_{u=1}^{\infty}\frac{\mu(u)}{u^{1+\varepsilon}}+\mathcal{O}(x)\ll\frac{x}{\varepsilon}+\mathcal{O}(x)\ll\frac{x}{\varepsilon}. \qquad\square$$

Note. When $\varepsilon=0$, the series in the right-term is formally equal to $\sum_{u=1}^{\infty}\frac{\mu(u)}{u}=0$ and we recall [1, p. 209] that

$$\sum_{u\le x}\frac{\mu(u)}{u}=\mathcal{O}\Big(\Big(\sqrt{\log x}\Big)\ e^{-c\sqrt{\log x}}\Big).$$

Obviously, $\frac{J_\varepsilon(d)}{d^\varepsilon}\to 0$ as $\varepsilon\to 0$. The summatory function $\sum_{1\le d\le x}\frac{J_\varepsilon(d)}{d^\varepsilon}$ naturally tightly depends on the smallness of ε.

4. The summatory function $\sum_{n\le x}\Big(\frac{J_\varepsilon(n)}{n^\varepsilon}\Big)^2$.

Theorem 4.1. *For $\varepsilon \ge 1/2$,*

$$\sum_{n\le x}\Big(\frac{J_\varepsilon(n)}{n^\varepsilon}\Big)^2 = \frac{x}{\zeta(1+\varepsilon)^2}\sum_{t\le x}\frac{|\mu(t)|\phi^{-1}(t)}{J^2_{1+\varepsilon}(t)} + \mathcal{O}_\varepsilon\Big(\frac{x^{1-\varepsilon}}{\varepsilon\,\zeta(1+\varepsilon)}\Big) + \mathcal{O}\big(\Delta_\varepsilon(x)\big),$$

where $\phi^{-1}(t)$ denotes the inverse of $\phi(t)$ and $\Delta_\varepsilon(x)$ is defined by (4.1).

Theorem 4.2.

$$x\,\exp\Big\{-c(\varepsilon)\frac{(\log x)^{1-\varepsilon}}{\log\log x}\Big\} \ll \sum_{1\le d\le x}\Big(\frac{J_\varepsilon(d)}{d^\varepsilon}\Big)^2 \ll x.$$

Proof of Theorem 4.1. As $\frac{J_\varepsilon(n)}{n^\varepsilon} = \sum_{u|n}\frac{\mu(u)}{u^\varepsilon}$, we have

$$\Big(\frac{J_\varepsilon(n)}{n^\varepsilon}\Big)^2 = \sum_{\substack{u|n\\ v|n}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon}.$$

Thus

$$\begin{aligned}\sum_{n\le x}\Big(\frac{J_\varepsilon(n)}{n^\varepsilon}\Big)^2 &= \sum_{n\le x}\sum_{\substack{u|n\\ v|n}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon} = \sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon}\Bigg(\sum_{\substack{n\le x\\ [u,v]|n}}1\Bigg)\\ &= \sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon}\Big(\frac{x}{[u,v]}+\mathcal{O}(1)\Big)\\ &= x\sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u,v]} + \mathcal{O}\Bigg(\Big(\sum_{u\le x}\frac{|\mu(u)|}{u^\varepsilon}\Big)^2\Bigg)\\ &= x\sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u,v]} + \mathcal{O}\Big(\Delta_\varepsilon(x)\Big),\end{aligned}$$

where

$$\Delta_\varepsilon(x) = \begin{cases} x^{2-2\varepsilon} & \text{for} \quad \frac{1}{2} \le \varepsilon < 1, \\ \log x & \text{for} \quad \varepsilon = 1, \\ 1 & \text{for} \quad \varepsilon > 1. \end{cases} \tag{4.1}$$

Now

$$\begin{aligned} \sum_{\substack{u \le x \\ v \le x}} \frac{\mu(u)\mu(v)}{(uv)^\varepsilon [u,v]} &= \sum_{d \le x} d \sum_{\substack{u \le x\,,\, v \le x \\ (u,v)=d}} \frac{\mu(u)\mu(v)}{(uv)^{1+\varepsilon}} \\ &= \sum_{d \le x} \frac{1}{d^{1+2\varepsilon}} \sum_{\substack{u' \le x/d\,,\, v' \le x/d \\ (u',v')=1}} \frac{\mu(u'd)\mu(v'd)}{(u'v')^{1+\varepsilon}} \\ &= \sum_{d \le x} \frac{1}{d^{1+2\varepsilon}} \sum_{u' \le x/d\,,\, v' \le x/d} \frac{\mu(u'd)\mu(v'd)}{(uv)^{1+\varepsilon}} \left(\sum_{\substack{w \mid u' \\ w \mid v'}} \mu(w) \right) \\ &= \sum_{d \le x} \frac{1}{d^{1+2\varepsilon}} \sum_{w \le x/d} \mu(w) \left(\sum_{\substack{u' \le x/d \\ w \mid u'}} \frac{\mu(u'd)}{(u')^{1+\varepsilon}} \right)^2 \\ &= \sum_{d \le x} \frac{1}{d^{1+2\varepsilon}} \sum_{w \le x/d} \mu(w) \left(d^{1+\varepsilon} \sum_{\substack{h \le x \\ dw \mid h}} \frac{\mu(h)}{h^{1+\varepsilon}} \right)^2 \\ &= \sum_{d \le x} d \sum_{w \le x/d} \mu(w) \left(\sum_{\substack{h \le x \\ dw \mid h}} \frac{\mu(h)}{h^{1+\varepsilon}} \right)^2 . \end{aligned} \tag{4.2}$$

We shall first estimate for a given integer θ the sum

$$\sum_{\substack{h \le x \\ \theta \mid h}} \frac{\mu(h)}{h^{1+\varepsilon}} ,$$

Let $\mathcal{P}_x$ denotes the collection of prime numbers less than x and $\mathcal{P} = \mathcal{P}_\infty$. If θ is not squarefree, then $\mu(h) = 0$ and this sum reduces to 0. Otherwise, let $\mathcal{Q}_x$ denote the set of integers θ such that

$$\theta = p_1 \dots p_r \,, \qquad \text{where} \qquad p_j \in \mathcal{P}_x \,,\ \ \theta \le x \,.$$

Only those integers h of type $h = p_1 \dots p_r \cdot q_1 \dots q_s$, with $q_i \in \mathcal{P}_x \backslash \{p_i\}$, contribute to the sum. We thereby get the sum

$$\frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \sum_{\substack{k \le x \\ (k,\theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}},$$

if $\theta \in \mathcal{Q}_x$, and this sum is 0 otherwise. Thus

$$\sum_{\substack{h \le x \\ \theta | h}} \frac{\mu(h)}{h^{1+\varepsilon}} = \begin{cases} \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \sum_{\substack{k \le x \\ (k,\theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}}, & \theta \in \mathcal{Q}_x, \\ 0, & \theta \notin \mathcal{Q}_x. \end{cases} \tag{4.3}$$

Let r be a squarefree positive integer. By formula p. 196 in [3], for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(nr)}{n^s} = \mu(r) \sum_{\substack{n=1 \\ (n,r)=1}}^{\infty} \frac{\mu(n)}{n^s} = \frac{\mu(r) r^s}{J_s(r)\zeta(s)}. \tag{4.4}$$

Thus

$$\begin{aligned} \sum_{\substack{h \le x \\ \theta | h}} \frac{\mu(h)}{h^{1+\varepsilon}} &= \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \left(\sum_{\substack{k=1 \\ (k,\theta)=1}}^{\infty} \frac{\mu(k)}{k^{1+\varepsilon}} - \sum_{\substack{k > x \\ (k,\theta)=1}} \frac{\mu(k)}{k^{1+\varepsilon}} \right) \\ &= \frac{\mu(\theta)}{\theta^{1+\varepsilon}} \cdot \left(\frac{\theta^{1+\varepsilon}}{J_{1+\varepsilon}(\theta)\zeta(1+\varepsilon)} + \mathcal{O}_\varepsilon(x^{-\varepsilon}) \right) \\ &= \frac{\mu(\theta)}{J_{1+\varepsilon}(\theta)\zeta(1+\varepsilon)} \\ &\quad + \mathcal{O}_\varepsilon(\theta^{-(1+\varepsilon)} x^{-\varepsilon}). \end{aligned} \tag{4.5}$$

In (4.2), $\theta = dw$ and thus must belong to $\mathcal{Q}_x$, in particular we must have $(d, w) = 1$. Thus

$$d | h, \quad w | h, \quad d, w \in \mathcal{Q}_x \quad \text{and} \quad (d, w) = 1.$$

Whence

$$\sum_{\substack{h \le x \\ dw | h}} \frac{\mu(h)}{h^{1+\varepsilon}} = \frac{\mu(d)\mu(w)}{J_{1+\varepsilon}(d) J_{1+\varepsilon}(w) \zeta(1+\varepsilon)} + \mathcal{O}_\varepsilon \left(\frac{1}{(dw)^{1+\varepsilon} x^\varepsilon} \right).$$

By reporting

$$\sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^{\varepsilon}[u,v]}=\sum_{d\le x}d\sum_{\substack{w\le x/d\\ (d,w)=1}}\mu(w)\left(\sum_{\substack{h\le x\\ dw|h}}\frac{\mu(h)}{h^{1+\varepsilon}}\right)^2$$

$$=\sum_{d\le x}d\sum_{\substack{w\le x/d\\ (d,w)=1}}\mu(w)\left(\frac{\mu(d)\mu(w)}{J_{1+\varepsilon}(d)J_{1+\varepsilon}(w)\zeta(1+\varepsilon)}+\mathcal{O}_{\varepsilon}\left(\frac{1}{(dw)^{1+\varepsilon}x^{\varepsilon}}\right)\right)^2$$

$$=\frac{1}{\zeta(1+\varepsilon)^2}\sum_{d\le x}\frac{d\mu(d)}{J^2_{1+\varepsilon}(d)}\sum_{\substack{w\le x/d\\ (d,w)=1}}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(w)}$$

$$+\mathcal{O}_{\varepsilon}\left(\frac{1}{\zeta(1+\varepsilon)x^{\varepsilon}}\sum_{d\le x}\frac{1}{d^{\varepsilon}J_{1+\varepsilon}(d)}\sum_{\substack{w\le x/d\\ (d,w)=1}}\frac{1}{w^{1+\varepsilon}J_{1+\varepsilon}(w)}\right)$$

$$+\mathcal{O}_{\varepsilon}\left(\frac{1}{x^{2\varepsilon}}\sum_{d\le x}\frac{1}{d^{1+2\varepsilon}}\sum_{\substack{w\le x/d\\ (d,w)=1}}\frac{1}{w^{2(1+\varepsilon)}}\right)$$

$$=\frac{1}{\zeta(1+\varepsilon)^2}\sum_{d\le x}\frac{d\mu(d)}{J^2_{1+\varepsilon}(d)}\sum_{\substack{w\le x/d\\ (d,w)=1}}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(w)}$$

$$+\mathcal{O}_{\varepsilon}\left(\frac{1}{\varepsilon\,\zeta(1+\varepsilon)x^{\varepsilon}}\right)+\mathcal{O}_{\varepsilon}\left(\frac{1}{\varepsilon\,x^{2\varepsilon}}\right)$$

$$=\frac{1}{\zeta(1+\varepsilon)^2}\sum_{w\le x}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(w)}\sum_{\substack{d\le x/w\\ (d,w)=1}}\frac{d\mu(d)}{J^2_{1+\varepsilon}(d)}$$

$$+\mathcal{O}_{\varepsilon}\left(\frac{1}{\varepsilon\,\zeta(1+\varepsilon)x^{\varepsilon}}\right). \tag{4.6}$$

For the main term we have

$$\frac{1}{\zeta(1+\varepsilon)^2}\sum_{w\le x}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(w)}\sum_{\substack{d\le x/w\\ (d,w)=1}}\frac{d\mu(d)}{J^2_{1+\varepsilon}(d)}=\frac{1}{\zeta(1+\varepsilon)^2}\sum_{t\le x}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(t)}\sum_{\substack{w\le x,d\le x/w\\ (d,w)=1,dw=t}}d\mu(d)$$

$$=\frac{1}{\zeta(1+\varepsilon)^2}\sum_{t\le x}\frac{|\mu(t)|}{J^2_{1+\varepsilon}(t)}\sum_{\substack{d\le x,\\ d|t}}d\mu(d),$$

where we have used $|\mu(d)| = \mu(d)^2$, $\mu(w)|\mu(d)| = \mu(w)^3|\mu(d)| = |\mu(wd)|\mu(w)$ for $(w,d)=1$; and $J_\varepsilon(n)$ is multi-function. Further the inner sum

$$\sum_{\substack{d\le x\\ d|t}} d\mu(d)$$

simplifies as follows

$$\sum_{d|t} d\mu(d) = \phi^{-1}(t),$$

since $t \le x$. Thus

$$\frac{1}{\zeta(1+\varepsilon)^2}\sum_{w\le x}\frac{|\mu(w)|}{J^2_{1+\varepsilon}(w)}\sum_{\substack{d\le x/w\\ (d,w)=1}}\frac{d\mu(d)}{J^2_{1+\varepsilon}(d)} = \frac{1}{\zeta(1+\varepsilon)^2}\sum_{t\le x}\frac{|\mu(t)|\phi^{-1}(t)}{J^2_{1+\varepsilon}(t)}.$$

So that

$$\sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u,v]} = \frac{1}{\zeta(1+\varepsilon)^2}\sum_{t\le x}\frac{|\mu(t)|\phi^{-1}(t)}{J^2_{1+\varepsilon}(t)} + \mathcal{O}_\varepsilon\left(\frac{1}{\varepsilon\,\zeta(1+\varepsilon)x^\varepsilon}\right).$$

Therefore

$$\begin{aligned}\sum_{n\le x}\left(\frac{J_\varepsilon(n)}{n^\varepsilon}\right)^2 &= x\sum_{\substack{u\le x\\ v\le x}}\frac{\mu(u)\mu(v)}{(uv)^\varepsilon[u,v]} + \mathcal{O}\big(\Delta_\varepsilon(x)\big)\\ &= \frac{x}{\zeta(1+\varepsilon)}\sum_{t\le x}\frac{|\mu(t)|\phi^{-1}(t)}{J^2_{1+\varepsilon}(t)} + \mathcal{O}_\varepsilon\left(\frac{x^{1-\varepsilon}}{\varepsilon\,\zeta(1+\varepsilon)}\right) + \mathcal{O}\big(\Delta_\varepsilon(x)\big).\end{aligned}$$

This achieves the proof. □

P r o o f o f T h e o r e m 4.2. Theorem 4.2 immediately follows from Lemma 3.1 and Theorem 4.1. □

5. The summatory function $\sum_{n=1}^{N}\frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}}$, $0<\theta<1$.

Theorem 5.1. *Let θ,ε be positive reals such that $0<\theta+\varepsilon<1$. Then,*

$$\sum_{n=1}^{N}\frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} \le C_{-\theta}\left(N^{1-\theta}\left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^\varepsilon}\right) + N^{1-\theta-\varepsilon}(\log N)^2\right),$$

where $C_{-\theta}$ is defined in (5.3) and

$$H(\varepsilon) = \sum_p \frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}}.$$

$$\sum_{n=1}^{N} \frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} \le C_{1-\theta}\left(N^{1-\theta}\left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^\varepsilon}\right) + N^{1-\theta-\varepsilon}(\log N)^2\right). \quad (5.1)$$

Proposition 5.2. *We have*

$$H(\varepsilon) = \frac{3}{\log 2}\cdot\frac{1}{\varepsilon} - \log\left(\frac{1}{\varepsilon}\right) + \mathcal{O}(1), \qquad \text{as } \varepsilon \to 0.$$

Proposition applies notably when ε is small, depending on N, the typical range of values being given by the interval $[1/\log N, 1/\log\log N]$.

P r o o f o f T h e o r e m 5.1. Use the simple notation $P = P(p) = \frac{p^\varepsilon}{2-p^{-\varepsilon}}$, and note that $1 - P^{-1} = (1-p^{-\varepsilon})^2$.

Define an auxiliary (completely multiplicative) arithmetical function h as follows,

$$h(p^\alpha) = P^\alpha, \qquad h(n) = \prod_{p^\alpha || n} P^\alpha. \quad (5.2)$$

By arguing as in [4, p. 36],

$$\frac{J_\varepsilon^2(n)}{n^{2\varepsilon}} = \prod_{p|n}(1-p^{-\varepsilon})^2 = \prod_{p|n}(1-P^{-1}) = \sum_{u|n}\frac{\mu(u)}{h(u)}.$$

Besides,

$$\sum_{n=1}^{N}\frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} = \sum_{n=1}^{N}\frac{1}{n^\theta}\sum_{d|n}\frac{\mu(d)}{h(d)} = \sum_{d\le N}\frac{\mu(d)}{h(d)}\sum_{\substack{n=1\\ d|n}}^{N}\frac{1}{n^\theta}$$

$$= \sum_{d\le N}\frac{\mu(d)}{h(d)d^\theta}\sum_{m=1}^{N/d}\frac{1}{m^\theta}.$$

Let $-1 < \alpha < 0$. Then for $M \ge 1$,

$$\sum_{k=1}^{M} k^\alpha = \frac{M^{\alpha+1}}{\alpha+1} + \mathcal{O}\big((M^\alpha)\big) + \left(\frac{1}{2} - \frac{1}{\alpha+1}\right)$$

$$+ \alpha(\alpha-1)\sum_{k=1}^{\infty}\int_0^1 \frac{t-t^2}{2}(k+t)^{\alpha-2}\,\mathrm{d}t - \sum_{k=M}^{\infty}\mathcal{O}\left(k^{\alpha-2}\right)$$

$$= \frac{M^{\alpha+1}}{\alpha+1} + C_\alpha + \mathcal{O}\left(M^\alpha\right),$$

where

$$C_\alpha = \frac{1}{2} - \frac{1}{\alpha+1} + \alpha(\alpha-1)\sum_{k=1}^{\infty}\int_0^1 \frac{t-t^2}{2}(k+t)^{\alpha-2}\,\mathrm{d}t\,. \tag{5.3}$$

Thus

$$\sum_{m=1}^{N/d}\frac{1}{m^\theta} = \frac{1}{1-\theta}\left(\frac{N}{d}\right)^{1-\theta} + C_{-\theta} + \mathcal{O}\left(\left(\frac{N}{d}\right)^{-\theta}\right).$$

Whence,

$$\sum_{n=1}^{N}\frac{J_\varepsilon^2(n)}{n^{\theta+2\varepsilon}} = \frac{N^{1-\theta}}{1-\theta}\sum_{d\le N}\frac{\mu(d)}{h(d)d} + C_{-\theta}\sum_{d\le N}\frac{\mu(d)}{h(d)d^\theta}$$

$$+\ \mathcal{O}\left(N^{-\theta}\sum_{d\le N}\frac{|\mu(d)|}{h(d)}\right). \tag{5.4}$$

We first estimate the sum $\sum_{d\le N}\frac{\mu(d)}{h(d)d}$. Now

$$\sum_{d=1}^{\infty}\frac{\mu(d)}{h(d)d} = \prod_p\left(1-\frac{1}{h(p)p}\right) = \prod_p\left(1-\frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}}\right)$$

Thus $0<\sum_{n=1}^{\infty}\frac{\mu(d)}{h(d)d}<\infty$ and

$$\sum_{d=1}^{\infty}\frac{\mu(d)}{h(d)d} = \exp\left\{\sum_p\left(1-\frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}}\right)\right\} \le \exp\left\{-\sum_p\frac{(2-p^{-\varepsilon})}{p^{1+\varepsilon}}\right\}$$
$$= \exp\{-H(\varepsilon)\}.$$

We have

$$h(n) = \prod_{p^\alpha||n}P^\alpha = \prod_{p^\alpha||n}\left(\frac{p^\varepsilon}{2-p^{-\varepsilon}}\right)^\alpha \ge \frac{n^\varepsilon}{2^{\Omega(n)}},$$

where $\Omega(n)$ is the sum of prime divisor function. Thus

$$\sum_{d>N}\frac{1}{h(d)d} \le \sum_{d>N}\frac{2^{\Omega(d)}}{d^{1+\varepsilon}}.$$

We use the fact that

$$\sum_{n\le x}2^{\Omega(n)} = C_0x(\log x)^2 + \mathcal{O}(x\log x),$$

where C_0 is numerical. Applying Abel summation gives,

$$\sum_{m=N+1}^{N+M}\frac{2^{\Omega(d)}}{d^{1+\varepsilon}} \le C\left\{\sum_{\mu=N+1}^{N+M}\frac{(\log\mu)^2}{\mu^{1+\varepsilon}} + \frac{(\log(M+N))^2}{(M+N)^\varepsilon}\right\}.$$

Hence, by letting M tend to infinity,

$$\sum_{d>N} \frac{1}{h(d)d} \le C \sum_{\mu>N} \frac{(\log \mu)^2}{\mu^{1+\varepsilon}} \le \frac{C}{\varepsilon N^{\varepsilon}}.$$

We deduce that

$$\left| \sum_{d\le N} \frac{\mu(d)}{h(d)d} \right| \le \exp\{-H(\varepsilon)\} + \frac{C}{\varepsilon N^{\varepsilon}}.$$

Consider now the second sum $\sum_{d\le N} \frac{\mu(d)}{h(d)d^{\theta}}$ in the right-term of (5.4), we have by using Abel summation again,

$$\sum_{d\le N} \frac{1}{h(d)d^{\theta}} \le \sum_{d\le N} \frac{2^{\Omega(d)}}{d^{\theta+\varepsilon}} \le C_{\theta,\varepsilon} N^{1-\theta-\varepsilon} (\log N)^2.$$

Finally, the third sum $\sum_{d\le N} \frac{\mu(d)}{h(d)}$ is special case ($\theta = 0$) of the previous and we have,

$$N^{-\theta} \sum_{d\le N} \frac{|\mu(d)|}{h(d)} \le N^{-\theta} \sum_{d\le N} \frac{1}{h(d)} \le C_{\theta,\varepsilon} N^{1-\theta-\varepsilon} (\log N)^2.$$

By combining these estimates, we get

$$\sum_{n=1}^{N} \frac{J_{\varepsilon}^2(n)}{n^{\theta+2\varepsilon}} \le C_{-\theta} \left(N^{1-\theta} \left(\exp\{-H(\varepsilon)\} + \frac{1}{\varepsilon N^{\varepsilon}} \right) + N^{1-\theta-\varepsilon} (\log N)^2 \right). \quad \square$$

Proof of Proposition 5.2. Set

$$A(x) = \sum_{n\le x} a(n), f(x) = \frac{1}{x^{1+\varepsilon}},$$

and $a(n) = 1$ for n is prime, otherwise $a(n) = 0$. Then we have

$$A(x) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x} \right).$$

Applying Abel's identity,

$$\begin{aligned} \sum_{p\le x} \frac{1}{p^{1+\varepsilon}} &= \sum_{n\le x} \frac{a(n)}{n^{1+\varepsilon}} = A(x)f(x) - \int_2^x A(t)f'(t)\,\mathrm{d}t \\ &= \frac{1}{x^{\varepsilon}\log x} + \mathcal{O}\left(\frac{1}{x^{\varepsilon}\log^2 x} \right) + (1+\varepsilon) \int_2^x \frac{1}{t^{1+\varepsilon}\log t} \\ &\quad + \mathcal{O}\left(\frac{1}{t^{1+\varepsilon}\log^2 t} \right) \mathrm{d}t. \end{aligned} \tag{5.5}$$

We have

$$\begin{aligned}\int_2^x \frac{1}{t^{1+\varepsilon}\log t}\,\mathrm{d}t &= \frac{1}{\log 2}\int_2^x \frac{1}{t^{1+\varepsilon}}\,\mathrm{d}t + \int_2^x \frac{1}{t^{1+\varepsilon}}\left(\frac{1}{\log t}-\frac{1}{\log 2}\right)\mathrm{d}t\\
&= \frac{1}{\varepsilon\log 2}(2^{-\varepsilon}-x^{-\varepsilon}) - \int_2^x \frac{1}{t^{1+\varepsilon}}\int_{\log 2}^{\log t}\frac{1}{u^2}\,\mathrm{d}u\,\mathrm{d}t\\
&= \frac{1}{\varepsilon\log 2}(2^{-\varepsilon}-x^{-\varepsilon}) - \int_{\log 2}^{\log x}\frac{1}{u^2}\,du\int_{e^u}^x \frac{1}{t^{1+\varepsilon}}\,\mathrm{d}t\\
&= \frac{1}{\varepsilon\log 2}(2^{-\varepsilon}-x^{-\varepsilon}) + \frac{1}{\varepsilon}\int_{\log 2}^{\log x}\frac{1}{u^2}(x^{-\varepsilon}-e^{-u\varepsilon})\,\mathrm{d}u\\
&= \frac{1}{\varepsilon\log 2}(2^{-\varepsilon}-x^{-\varepsilon}) + \frac{x^{-\varepsilon}}{\varepsilon}\left(\frac{1}{\log 2}-\frac{1}{\log x}\right) + \frac{1}{\varepsilon}\int_{\log 2}^{\log x}\frac{e^{-u\varepsilon}}{u^2}\,\mathrm{d}u,
\end{aligned}\tag{5.6}$$

and for $\varepsilon \geq \frac{1}{\log x}$,

$$\begin{aligned}\int_{\log 2}^{\log x}\frac{e^{-u\varepsilon}}{u^2}\,\mathrm{d}u &= \int_{\log 2}^{\frac{1}{\varepsilon}}\frac{e^{-u\varepsilon}}{u^2}\,\mathrm{d}u + \int_{\frac{1}{\varepsilon}}^{\log x}\frac{e^{-u\varepsilon}}{u^2}\,\mathrm{d}u\\
&= \int_{\log 2}^{\frac{1}{\varepsilon}}\frac{1}{u^2}\left(1-u\varepsilon+\mathcal{O}(u^2\varepsilon^2)\right)\,\mathrm{d}u + \mathcal{O}\left(\varepsilon^2\int_{\frac{1}{\varepsilon}}^{\log x} e^{-u\varepsilon}\,\mathrm{d}u\right)\\
&= \frac{1}{\log 2} + (\log\log 2)\varepsilon - (\varepsilon-(\log\varepsilon)\varepsilon) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\varepsilon x^{-\varepsilon}\right),
\end{aligned}\tag{5.7}$$

for $\varepsilon \leq \frac{1}{\log x}$,

$$\begin{aligned}\int_{\log 2}^{\log x}\frac{e^{-u\varepsilon}}{u^2}\,\mathrm{d}u &= \int_{\log 2}^{\log x}\frac{1}{u^2}\Big(1-u\varepsilon+\mathcal{O}(u^2\varepsilon^2)\Big)\,\mathrm{d}u\\
&= \frac{1}{\log 2} + (\log\log 2)\varepsilon - \left(\frac{1}{\log x} + (\log\log x)\varepsilon\right) + \mathcal{O}(\varepsilon)\\
&= \frac{1}{\log 2} + (\log\log 2)\varepsilon - \frac{1}{\log x}\\
&\quad + \mathcal{O}\left(\log\log x\right)\varepsilon + \mathcal{O}(\varepsilon)\\
&= \frac{1}{\log 2} + (\log\log 2)\varepsilon - \frac{1}{\log x} + \mathcal{O}\left(\frac{1}{\log x}\log\log x\right) + \mathcal{O}(\varepsilon)\\
&= \frac{1}{\log 2} + (\log\log 2)\varepsilon + \mathcal{O}\left(\frac{1}{\log x}\log\log x\right) + \mathcal{O}(\varepsilon),
\end{aligned}\tag{5.8}$$

then by (5.5)–(5.8),

$$\begin{aligned}\sum_{p\le x}\frac{1}{p^{1+\varepsilon}} = \frac{1}{x^{\varepsilon}\log x} &+ \mathcal{O}\left(\frac{1}{x^{\varepsilon}\log^2 x}\right)\\ &+(1+\varepsilon)\left(\frac{1}{\varepsilon\log 2}(2^{-\varepsilon}-x^{-\varepsilon})+\frac{x^{-\varepsilon}}{\varepsilon}\left(\frac{1}{\log 2}-\frac{1}{\log x}\right)\right.\\ &+\frac{1}{\varepsilon\log 2}+(\log\log 2)-(1-\log\varepsilon)\\ &+\mathcal{O}\left(\frac{1}{\varepsilon\log x}\log\log x\right)+\mathcal{O}(1)+\mathcal{O}\left(x^{-\varepsilon}\right)\Bigg)\\ &+\mathcal{O}\left(\int_2^x\frac{1}{t^{1+\varepsilon}\log^2 t}\right)\mathrm{d}t. \end{aligned} \tag{5.9}$$

Let $x\to\infty$ in (5.9), thus

$$\sum_{p=1}^{\infty}\frac{1}{p^{1+\varepsilon}} = (1+\varepsilon)\left(\frac{2^{-\varepsilon}}{\varepsilon\log 2}+\frac{1}{\varepsilon\log 2}-(1-\log\varepsilon)\right)+\mathcal{O}(1). \tag{5.10}$$

For $\varepsilon\to 0$, (5.10) yields

$$\sum_{p=1}^{\infty}\frac{1}{p^{1+\varepsilon}} = \frac{2}{\varepsilon\log 2}-\log\left(\frac{1}{\varepsilon}\right)+\mathcal{O}(1). \tag{5.11}$$

By (5.11),

$$\begin{aligned} H(\varepsilon)=\sum_{p=1}^{\infty}\frac{2-p^{-\varepsilon}}{p^{1+\varepsilon}} &= 2\left(\frac{2}{\varepsilon\log 2}-\log\left(\frac{1}{\varepsilon}\right)+\mathcal{O}(1)\right)\\ &-\left(\frac{1}{\varepsilon\log 2}-\log\left(\frac{1}{\varepsilon}\right)+\mathcal{O}(1)\right)\\ &=\frac{3}{\log 2}\cdot\frac{1}{\varepsilon}-\log\left(\frac{1}{\varepsilon}\right)+\mathcal{O}(1). \end{aligned}$$

□

REFERENCES

[1] BATEMAN, P.—DIAMOND, H. G.: *Analytic Number Theory—An Introductory Course.* Monographs in Number Theory, Vol. 1, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2004.

[2] BALASUBRAMANIAN, R.—LUCA, F.—RALAIVAOSAONA, D.: *On the sum of the first n values of the Euler function*, Acta Arith. **163** (2014), no. 3, 199–201.

[3] MCCARTHY, P. J.: *Introduction to Arithmetical Functions*. (Universitext), Springer--Verlag, New-York Inc., 1986.

[4] MONTGOMERY, H.—VAUGHAN, R.: *Multiplicative Number Theory: I. Classical Theory*. Cambridge Stud. Adv. Math. Vol. 97, Cambridge University Press, Cambridge, 2007

[5] SHAPIRO, H. N.: *On a theorem of Selberg and generalization*, Ann. of Math. **51** 1950), no. 2. 485–497.

[6] TITCHMARSH, E. C.: *The Theory of the Riemann-Zeta function*. Second Edition (Edited and with a preface by D. R. Heath-Brown.) The Clarendon Press, Oxford University Press, New York, 1986.

[7] WILSON, B. M.: *Proofs of some formulae enunciated by Ramanujan*, Proc. London Math. Soc. **21** (1923), no. 2, 235–255.

Received November 11, 2023
Accepted December 20, 2023

School of Mathematics.
Hefei University of Technology
Hefei 230009
P. R. CHINA

E-mail: vera123_99@hotmail.com
michel.weber@math.unistra.fr
m.j.g.weber@mailo.com