sciendo

uniform distribution theory

DOI: 10.2478/UDT-2022-0016
Unif. Distrib. Theory **17** (2022), no.2, 161–164

DISTRIBUTION OF LEADING DIGITS OF IMAGINARY PARTS OF RIEMANN ZETA ZEROS

Yukio Ohkubo[1] — Oto Strauch[2]

[1]Department of Business Administration, The International University of Kagoshima, Sakanoue, Kagoshima-shi, JAPAN

[2]Mathematical Institute, Slovak Academy of Sciences, Bratislava, SLOVAKIA

ABSTRACT. In this paper we study the distribution of leading digits of imaginary parts of Riemann zeta zeros in the b-adic expansion.

Communicated by Vladimír Baláž

Introduction

Throughout this paper, for $x \in \mathbb{R}$, let $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$, $\{x\} = x - [x] = x \bmod 1$. For fixed sequence of real numbers $x_1, x_2, \dots x_n, \dots$ and arbitrary integer sequence $N_1 < N_2 < \cdots < N_i < \cdots$ denote

$$F_{N_i}(x) = \frac{\#\{n \leq N_i; x_n \bmod 1 \in [0, x)\}}{N_i}.$$

The set of all possible limits $g(x) = \lim_{N_i \to \infty} F_{N_i}(x)$ for continuity points x of $g(x)$ is called $G(x_n \bmod 1)$ — the set of all distribution functions $g(x)$ of the real sequence $x_n \bmod 1$.

Let $b \geq 2$ be an integer considered to be the base of the numeral system used for the representation of a positive real number $x > 0$ and $M_b(x)$ be a mantissa of x defined by $x = M_b(x) \times b^{n(x)}$ such that $1 \leq M_b(x) < b$ holds, where $n(x)$ is a uniquely determined integer. Let $K = k_1 k_2 \dots k_r$ be a positive integer expressed

2020 Mathematics Subject Classification: 11K06, 11K31, 11K38.
Keywords: Benford's law, distribution function, zeros of Riemann zeta function.
Supported by Grant VEGA no. 2/0119/23.

in the base b, that is

$$K = k_1 \times b^{r-1} + k_2 \times b^{r-2} + \cdots + k_{r-1} \times b + k_r,$$

where $k_1 \neq 0$ and at the same time $K = k_1 k_2 \dots k_r$ is considered to be an $r-$consecutive block of integers in the base b.

Qualitative result

It is clear that the following basic equivalences hold.

$$\begin{aligned} & K \leq M_b(x) \times b^{r-1} < K+1 \\ \iff & \frac{K}{b^{r-1}} \leq M_b(x) < \frac{K+1}{b^{r-1}} \qquad (1) \\ \iff & \log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b\big(M_b(x)\big) < \log_b\left(\frac{K+1}{b^{r-1}}\right) \\ \iff & \log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b x \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right). \qquad (2) \end{aligned}$$

Note that for x of the type $x = 0.00\dots0k_1k_2\dots k_r\dots$, we shall omit the first zero digits and $M_b(x) = k_1.k_2\dots k_r\dots$ Let

$$x_n > 0, n = 1, 2, \dots, g(x) \in G\big(\log_b(x_n) \bmod 1\big) \quad \text{and} \quad g(x) = \lim_{i\to\infty} F_{N_i}(x),$$

where

$$F_{N_i}(x) = \frac{\#\{n \leq N_i; \log_b x_n \bmod 1 \in [0, x)\}}{N_i}.$$

Using equivalent inequalities (1) and (2), then for fixed N_i we have

$$\begin{aligned} & \frac{\#\big\{n \leq N_i;\ \text{first } r \text{ digits of } M_b(x_n) \text{ are equal to } K\big\}}{N_i} \\ &= \frac{\#\big\{n \leq N_i; \log_b\left(\frac{K}{b^{r-1}}\right) \leq \log_b x_n \bmod 1 < \log_b\left(\frac{K+1}{b^{r-1}}\right)\big\}}{N_i} \\ &= F_{N_i}\left(\frac{K+1}{b^{r-1}}\right) - F_{N_i}\left(\frac{K}{b^{r-1}}\right). \end{aligned}$$

For the following Benford's law we use Theorem 47, page 78, in [3].

Theorem 1. *Let $g(x) \in G(\log_b x_n \bmod 1)$ and $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. Then*

$$\lim_{N_i\to\infty} \frac{\#\{n \leq N_i;\ \textit{first } r \textit{ digits (starting with a non-zero digit) of } x_n = K\}}{N_i} = g\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - g\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right).$$

Let γ_n, $n = 1, 2, \dots$, be the sequence of all positive imaginary parts of non-trivial zeros of the Riemann zeta function $\zeta(s)$ in ascending order. For a positive integer N let

$$F_N(x) = \frac{1}{N}\#\{1 \le n \le N; \log_b \gamma_n \bmod 1 \in [0, x)\} \quad \text{for } 0 \le x \le 1.$$

We need find the set $G(\log_b \gamma_n \bmod 1)$ of all distribution functions of $\log_b \gamma_n \bmod 1$. To do this we use Kemperman's theorem [2].

Theorem 2. *Assume that*

$$\lim_{n\to\infty}\big(f(n+1) - f(n)\big) = 0,$$
$$\lim_{n\to\infty} n\big(f(n+1) - f(n)\big) = t.$$

Then the set of all distribution functions $G(f(n) \bmod 1)$ *has the form*

$$g_u(x) = \begin{cases} \frac{e^{(1+x-u)/t} - e^{(1-u)/t}}{e^{1/t}-1} & \text{if } 0 \le x \le u, \\ 1 - \frac{e^{(1-u)/t} - e^{(x-u)/t}}{e^{1/t}-1} & \text{if } u \le x \le 1. \end{cases}$$

Furthermore, $F_{N_i}(x) \to g_u(x)$ *if and only if* $f(N_i \bmod 1) \to u$. *In this case*

$$F_{N_i}(x) = \frac{\#\{n \le N_i; f(x_n) \bmod 1 \in [0, x)\}}{N_i}.$$

For the sequence $\gamma_n, n = 1, 2, \dots$ (see M. Hassani [1]) we have

$$\frac{2\pi n}{\log n}\left(1 + \frac{11}{12} \cdot \frac{\log\log n}{\log n}\right) \le \gamma_n \le \frac{2\pi n}{\log n}\left(1 + \frac{23}{12} \cdot \frac{\log\log n}{\log n}\right). \tag{3}$$

Using Theorem 11 page 25 in [3] we have

Theorem 3. *Let* x_n *and* y_n *be two real sequences. Assume that all distribution functions in* $G(x_n \bmod 1)$ *are continuous at* 0 *and* 1. *Then the zero limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n| = 0 \quad \textit{implies} \quad G(x_n \bmod 1) = G(y_n \bmod 1).$$

From Theorem 3 and formula (3) it follows that

$$G\left(\log_b \frac{2\pi n}{\log n} \bmod 1\right) = G(\log_b \gamma_n \bmod 1).$$

For Benford's law we need find distribution functions of the sequence $\log_b \gamma_n \bmod 1$. To do this we find distribution functions $f(n) = \log_b \frac{2\pi n}{\log n} \bmod 1$ using Kemperman's theorem 2. Here

$$\lim_{n\to\infty}\left(\log_b\left(\frac{2\pi(n+1)}{\log(n+1)}\right) - \log_b\left(\frac{2\pi n}{\log n}\right)\right) = 0,$$
$$\lim_{n\to\infty} n\left(\log_b\left(\frac{2\pi(n+1)}{\log(n+1)}\right) - \log_b\left(\frac{2\pi n}{\log n}\right)\right) = \frac{1}{\log b},$$

Thus from Theorem 2 we have

$$g_u(x) = \begin{cases} \frac{b^{(1+x-u)}-b^{1-u}}{b-1} & \text{if } 0 \le x \le u, \\ 1 - \frac{b^{(1-u)}-b^{x-u}}{b-1} & \text{if } u \le x \le 1. \end{cases} \tag{4}$$

THEOREM 4.

$$\lim_{N_i\to\infty} \frac{\#\{n \le N_i; \textit{first } r \textit{ digits (starting with a non-zero digit) of } \gamma_n = K\}}{N_i} = g_u\left(\log_b\left(\frac{K+1}{b^{r-1}}\right)\right) - g_u\left(\log_b\left(\frac{K}{b^{r-1}}\right)\right),$$

where

$$u = \lim_{N_i\to\infty} \log_b \frac{2\pi N_i}{\log N_i} \bmod 1.$$

Note that the equation (4) is equivalent to

$$g_u(x) = \frac{1}{b^u}\frac{b^x-1}{b-1} + \frac{b^{\min(x,u)}-1}{b^u}.$$

REFERENCES

[1] HASSANI, M.: *On the imaginary part of the nontrivial zeros of the Riemann zeta function*, (Pardalos, Panos M. ed. et al.) In:*Contributions in Mathematics and Engineering*. In honor of Constantin Carathéodory. With a foreword by R. Tyrrell Rockafellar. Springer,[Cham] Switzerland, 2017, pp. 313–321.

[2] KEMPERMAN, J. H. B.: *Distribution modulo 1 of slowly changing sequences*, Nieuw. Arch. Wisk. **21** (1973), no. 3 138–163.

[3] STRAUCH, O.: *Distribution of Sequences: A Theory*. VEDA, Bratislava, 2019, Prague, 2019,

Received June 17, 2022
Accepted October 12, 2022

Yukio Ohkubo
Department of Business Administration
The International University of Kagoshima
8-34-1 Sakanoue
Kagoshima-shi, 891-0197
JAPAN
E-mail: ohkubo@eco.iuk.ac.jp

Oto Strauch
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA
E-mail: strauch@mat.savba.sk