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uniform distribution theory

SUM OF DIVISORS OF FIBONACCI NUMBERS

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ABSTRACT. In this note, we prove an estimate on the count of Fibonacci numbers whose sum of divisors is also a Fibonacci number. As a corollary, we find that the series of reciprocals of indices of such Fibonacci numbers is convergent.

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1. Introduction

For a positive integer n, we write $\sigma(n)$ for the sum of divisors function of n. Recall that a number n is called *multiply perfect* if $n \mid \sigma(n)$. If $\sigma(n) = 2n$, then n is called *perfect.* Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers. In [4], it was shown that there are only finitely many multiply perfect Fibonacci numbers, and in [5], it was shown that no Fibonacci number is perfect. For a positive integer n, the value $\varphi(n)$ of the Euler function is defined to be the number of natural numbers less than or equal to n and coprime to n. In [6], it was shown that if $\varphi(F_n) = F_m$ then $n \in \{1, 2, 3, 4\}$.

In [7], Fibonacci numbers F_n with the property that the sum of their aliquot parts is also a Fibonacci number were investigated. This reduces to studying those positive integers n such that $\sigma(F_n) = F_n + F_m$ holds with some positive integers m. In [7], it was shown that such positive integers form a set of asymptotic density zero.

Here, we search for Fibonacci numbers F_n such that $\sigma(F_n)$ is a Fibonacci number. We put

$$\mathcal{A} = \{n : \sigma(F_n) = F_m \text{ for some positive integer } m\}.$$

For a positive real number x and a subset $\mathcal{B}$ of the positive integers, we write $\mathcal{B}(x) = \mathcal{B} \cap [1, x]$. In this note, we prove the following result.

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Theorem 1. *There are constants c_0 and C_0 such that inequality*

$$\#\mathcal{A}(x) < \frac{C_0 x \log\log\log x}{(\log x)^2}$$

holds for all $x > c_0$.

By partial summation, Theorem 1 immediately implies that

Corollary 1.1. *The series*

$$\sum_{n\in\mathcal{A}} \frac{1}{n}$$

is convergent.

We remark that it is quite possible that $\mathcal{A}\setminus\{1,2,3\}$ is empty, as computer searches for $n \le 5\cdot 10^3$ failed to find any other element of $\mathcal{A}$. The presumably larger set $\mathcal{B} = \{n : \sigma(n) = F_m$ for some positive integer $m\}$ contains the integers $1, 2, 7, 9, 66, 70, 94, 115, 119, 2479$. It is likely that $\mathcal{B}$ is infinite, but this is probably hard to prove.

Throughout this paper, we use the Vinogradov symbols $\gg$, $\ll$ and the Landau symbols O, $\asymp$ and o with their usual meanings. We recall that $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent and mean that $|A| < cB$ holds with some constant c, while $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold. For a positive real number x we write $\log x$ for the maximum between 1 and the natural logarithm of x. We use p, q, P and Q to denote prime numbers.

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2. The Proof

Let x be a large positive real number and assume that $n \le x$. We also assume that $n > x/(\log x)^2$, since there are at most $x/(\log x)^2$ positive integers failing this property.

2.1. The size of m in terms of n

It is known that $\sigma(n)/n \ll \log\log n$ (see Theorem 323 in Chapter 18 of [3]). Let $\gamma = (1+\sqrt{5})/2$ be the golden section. Since $F_n \asymp \gamma^n$, we get that

$$\gamma^{m-n} \ll \frac{F_m}{F_n} = \frac{\sigma(F_n)}{F_n} \ll \log\log F_n \ll \log n \le \log x,$$

therefore

$$m - n < c_1 \log\log x$$

holds for all sufficiently large values of x, where we can take $c_1 = 3$. From now on, we write $m = n + k$, where $k < K = \lfloor c_1 \log\log x \rfloor$.

2.2. Discarding smooth integers

Let $P(n)$ be the largest prime factor of n. Let

$$y = \exp\left(\frac{\log x \log\log\log x}{3\log\log x}\right).$$

Let

$$\mathcal{A}_1(x) = \{n \le x \ : \ P(n) \le y\}. \tag{1}$$

The elements of the set $\mathcal{A}_1(x)$ are refereed to as *y-smooth* numbers. By known results from the distribution of smooth numbers (see, for example, Chapter III.5 from [8]),

$$\#\mathcal{A}_1(x) \le x \exp\left(-(1+o(1))u \log u\right),$$

where $u = \log x/\log y$. In our case, we have $u = 3\log\log x/\log\log\log x$, therefore $u \log u = 3(1+o(1))\log\log x$, leading to

$$\#\mathcal{A}_1(x) \le \frac{x}{(\log x)^{3+o(1)}} < \frac{x}{(\log x)^2}, \tag{2}$$

once x is sufficiently large.

2.3. The order of apparition of $\sigma(F_{P(n)})$

For every positive integer n we write $z(n)$ for the *order of apparition of n in the Fibonacci sequence* which is defined as the smallest positive integer u such that $n \mid F_u$. It is known [2] that if $n \mid F_t$, then $z(n) \mid t$, and that $z(n) \gg \log n$.

Let $n \le x$ be not in $\mathcal{A}_1(x)$. Let $p = P(n)$ be its largest prime factor. Then $F_p \mid F_n$. We now show that F_p and F_n/F_p are coprime. It is known [1, Prop. 2.1] that

$$\gcd\left(F_p, \frac{F_n}{F_p}\right) \Bigm| \frac{n}{p}.$$

If the greatest common divisor appearing above were not 1, then there would exist a prime $Q \mid F_p$ such that $Q \mid n/p$. However, for large y (hence, for large x), $Q \equiv \pm 1 \pmod p$, therefore $Q \geq 2p-1 > p$, and so it cannot divide n/p which is a p-smooth number. Thus, F_p and F_n/F_p are coprime, and by the multiplicative property of σ we get that $\sigma(F_p) \mid \sigma(F_n)$. Hence, $\sigma(F_p) \mid F_m$, leading to $z(\sigma(F_p)) \mid m$.

Fix p and $k = m - n$. Then $p \mid n$ and $z(\sigma(F_p)) \mid n + k$. Further, note that p cannot divide $z(\sigma(F_p))$, for if this were the case, then the above congruences would lead to $p \mid k$, which is impossible for large x since $0 < k \leq K < y < p$. Thus, we can apply the Chinese Remainder Lemma and conclude that n is in a certain arithmetic progression modulo $pz(\sigma(F_p))$. Let $n_{k,p}$ be the least positive term of this progression, and let

$$\mathcal{A}_{k,p}(x) = \{n_{k,p} + pz(\sigma(F_p))\lambda \ : \ \lambda > 0\} \cap [1, x].$$

It is clear that $\#\mathcal{A}_{k,p}(x) \leq \lfloor x/pz(\sigma(F_p))\rfloor \leq x/pz(\sigma(F_p))$, therefore if we write

$$\mathcal{A}_2(x) = \bigcup_{\substack{0<k\leq K\\ y\leq p\leq x}} \mathcal{A}_{k,p}(x), \tag{3}$$

then we have the bound

$$\#\mathcal{A}_2(x) \leq \sum_{0<k\leq K} \sum_{y\leq p\leq x} \frac{x}{pz(\sigma(F_p))} \ll xK \sum_{y\leq p} \frac{1}{p^2} \ll \frac{x\log\log x}{y}, \tag{4}$$

where in the above estimate we used the fact that

$$z(\sigma(F_p)) \gg \log(\sigma(F_p)) \geq \log(F_p) \gg p.$$

We put

$$\mathcal{A}_3(x) = \{n_{k,p} \ : \ k \in [1, K] \text{ and } p \in [y, x]\} \tag{5}$$

and study $\mathcal{A}_3(x)$. Let $L_1 = (\log x)^2$, $L = (\log x)/2$ put $z_1 = x/L_1, z = x/L$, and write

$$\mathcal{A}_3(x) = \mathcal{A}_4(x) \cup \mathcal{A}_5(x) \cup \mathcal{A}_6(x),$$

where

$$\begin{aligned}
\mathcal{A}_4(x) &= \mathcal{A}_3(x) \cap \{n \leq x \ : \ P(n) < z_1\},\\
\mathcal{A}_5(x) &= \mathcal{A}_3(x) \cap \{n \leq x \ : \ z_1 \leq P(n) < z\},\\
\mathcal{A}_6(x) &= \mathcal{A}_3(x) \cap \{n \leq x \ : \ z \leq P(n)\}.
\end{aligned}$$

Since elements of $\mathcal{A}_4(x)$ are uniquely determined by their largest prime factor (at most z_1) and $k \in [1, K]$, we get that

$$\#\mathcal{A}_4(x) \leq K\pi(z_1) \leq \frac{x(\log\log x)^2}{(\log x)^3} \tag{6}$$

once x is sufficiently large. We will show that

$$\#\mathcal{A}_5(x) \ll \frac{x \log\log\log x}{(\log x)^2} \tag{7}$$

and that $\mathcal{A}_6(x)$ is empty for large values of x which, together with estimates (2), (4) and (6), will complete the proof of the theorem.

2.4. The end of the proof

From now on until the end of the proof, n is a positive integer in $\mathcal{A}_5(x) \cup \mathcal{A}_6(x)$. Then $n = pa$, where $a \le L_1$. Thus, $F_a \mid F_n$. Put $A = F_n/F_a$ and note that every prime factor P of A has the property that $p \mid z(P)$. In what follows, we will estimate $\sigma(A)/A$. First of all

$$\frac{\sigma(A)}{A} \le \frac{A}{\varphi(A)} = \prod_{P \mid A} \left(1 + \frac{1}{P-1}\right) \le \prod_{d \mid a} \prod_{z(P)=pd} \left(1 + \frac{1}{P-1}\right). \tag{8}$$

For each fixed $d \mid a$, we have

$$\prod_{z(P)=pd} \left(1 + \frac{1}{P-1}\right) \le \exp\left(\sum_{z(P)=pd} \frac{1}{P-1}\right).$$

It is known (see, for example, [7]), that for each fixed positive integer t we have

$$\sum_{z(P)=t} \frac{1}{P-1} \ll \frac{\log\log t}{\varphi(t)}.$$

Hence,

$$\prod_{z(P)=pd} \left(1 + \frac{1}{P-1}\right) \le \exp\left(O\left(\frac{\log\log(pd)}{p\varphi(d)}\right)\right) = \exp\left(O\left(\frac{\log\log x}{p\varphi(d)}\right)\right). \tag{9}$$

Thus, multiplying estimates (9) over all the divisors d of a and using (8), we get

$$1 \le \frac{\sigma(A)}{A} \le \exp\left(O\left(\frac{\log\log x}{p} \sum_{d \mid a} \frac{1}{\varphi(d)}\right)\right) < \exp\left(\frac{(\log\log x)^2}{p}\right)$$

for large x, where we used the fact that

$$\sum_{d \mid a} \frac{1}{\varphi(d)} \ll \log\log a \sum_{d \mid a} \frac{1}{d} \le \frac{\sigma(a) \log\log L_1}{a} \ll (\log\log L_1)^2 = o(\log\log x)$$

as $x \to \infty$. Hence,

$$0 < \frac{\sigma(A)}{A} - 1 < \exp\left(\frac{(\log\log x)^2}{p}\right) - 1 \le \frac{2(\log\log x)^2}{p} \le \frac{2(\log\log x)^2}{z_1}, \tag{10}$$

where in the last inequality we used the fact that

$$\frac{(\log\log x)^2}{p} \le \frac{(\log\log x)^2}{z_1} = o(1)$$

as $x \to \infty$ together with the fact that the inequality $e^t - 1 < 2t$ holds for all sufficiently small positive values of t.

We will use that $\sigma(F_n)/F_n$ is close to $\sigma(F_a)/F_a$ since

$$\frac{\sigma(F_a)}{F_a} < \frac{\sigma(F_n)}{F_n} \le \frac{\sigma(F_a)}{F_a}\frac{\sigma(A)}{A}. \tag{11}$$

In particular,

$$\frac{\sigma(F_n)}{F_n} \ll \frac{\sigma(F_a)}{F_a}.$$

Therefore,

$$k = m - n \ll \log\left(\frac{\sigma(F_n)}{F_n}\right) \ll \log\left(\frac{\sigma(F_a)}{F_a}\right) \ll \log\log a \ll \log\log\log x.$$

Now we are ready to estimate $\#\mathcal{A}_5(x)$:

$$\#\mathcal{A}_5(x) \ll \pi(L)\log\log\log x.$$

This completes the proof of (7).

We now turn to the study of $\mathcal{A}_6(x)$. We have to show that $\mathcal{A}_6(x) = \emptyset$. Assume that $n \in \mathcal{A}_6(x)$. By (11),

$$\frac{\sigma(A)}{A} - 1 \ge \frac{F_m}{A\sigma(F_a)} - 1 = \frac{F_m F_a}{F_n \sigma(F_a)} - 1.$$

Writing $F_t = (\gamma^t - \delta^t)/(\gamma - \delta)$, where $\delta = (1-\sqrt{5})/2 = -1/\gamma$, we get easily that

$$\frac{F_m F_a}{F_n\sigma(F_a)} - 1 = \frac{\gamma^{m-n}F_a - \sigma(F_a)}{\sigma(F_a)} + O\left(\gamma^{-2n}\right). \tag{12}$$

Since γ is quadratic irrational, it follows that the inequality

$$|U\gamma - V| > \frac{c_3}{U}$$

holds for all positive integers U and V with some positive constant c_3. Since $\gamma^{m-n} = F_{m-n}\gamma + F_{m-n-1}$, it follows that

$$\begin{aligned} |\gamma^{m-n}F_a - \sigma(F_a)| &= |(F_{m-n}F_a)\gamma - (\sigma(F_a) - F_aF_{m-n+1})| \\ &\gg \frac{1}{F_{m-n}F_a} \gg \frac{1}{\gamma^{m-n+a}} \gg \frac{1}{\gamma^{2L}}. \end{aligned} \tag{13}$$

Since $n > x/(\log x)^2$, it follows from estimates (12) and (13) that the lower bound

$$\frac{\sigma(A)}{A} - 1 > \frac{1}{\gamma^{4L}} \tag{14}$$

holds for large x. Combining estimates (10) and (14), we get

$$\frac{x}{(\log x)^2} \le 2(\log\log x)^2\gamma^{4L} = 2(\log\log x)^2 x^{2\log\gamma}.$$

which is impossible for large x because $2\log\gamma < 1$. This completes the proof of the fact that $\mathcal{A}_6(x)$ is empty for large x.

3. Further Remarks

In this note, we proved that for almost all positive integers n, $\sigma(F_n)$ is not a Fibonacci number, and by the result from [7] the same is true for $\sigma(F_n) - F_n$. Recall that the *Zeckendorf decomposition* of the positive integer n is its representation

$$n = F_{m_1} + \cdots + F_{m_t},$$

where $0 < m_t < \cdots < m_1$ and $m_{i+1} - m_i \ge 2$ for all $i = 1, \ldots, t-1$. It is known [9] that such a representation always exists and up to identifying F_2 with F_1, it is also unique. Let $\ell(n) = t$ be the length of the Zeckendorf decomposition of n. We conjecture that $\ell(\sigma(F_n))$ tends to infinity with n on a set of asymptotic density 1 and we would like to leave this question for the reader. Note that our main result shows that $\ell(\sigma(F_n)) \ge 2$ holds for almost all n.

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