Bifurcation Analysis of Non-linear Differential Equations

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Contents

Chapter 1

Introduction

A bifurcation of a dynamical system occurs when the parameter value of a system changes such that it causes a sudden qualitative change in its behaviour. Bifurcations occur in both continuous systems and discrete systems. I will only consider bifurcations in discrete systems.

Before studying bifurcation I will start by analysing the stability of ordinary differential equations both linear and non-linear with an arbitrary constant and look at how this constant affects the stability of stationary points. I will then go on to study bifurcation theory.

In this project I will study the different types of bifurcations that can arise in ordinary differential equations, the necessary conditions for each type of bifurcation to occur and the normal form.

I will then go on to consider systems of ordinary differential equations. Systems of ordinary differential equations can usually only be solved analytically if the system is linear. I will look at the method for solving linear systems as well as looking at the method of linearisation so that I can solve non-linear systems analytically close to an equilibrium point. I will then consider systems of ordinary differential equations with a parameter and study Hopf bifurcation.

Chapter 2

Definitions

Autonomous equation: $\frac{dx}{dt} = f(x)$ is called an autonomous equation because the differential equation does not explicitly depend on the independent variable, here denoted t for time. This means that the equation is 'timeless', the solution will be the same regardless of when in time it occurs.

Equilibrium: A state in which the system does not change with time, in particular the state variables remain constant.

Trajectory: A curve traced by the solution of a differential equation.

Phase portrait: A geometric representation of the set of trajectories of a dynamical system in the phase plane.

Non-hyperbolic: An equilibrium point x^* of $\frac{df}{dx} = f(x)$ is called non-hyperbolic if $\frac{\partial f}{\partial x}(x^*) = 0$.

Bifurcation: A bifurcation of a dynamical system is a qualitative change in its dynamics produced by varying parameters.

Chapter 3

Ordinary Differential Equations

Ordinary differential equations are functions of one independent variable and its derivatives. Here I will look at how to find the stability of equilibrium point of ODEs and how to determine the equilibrium's stability.

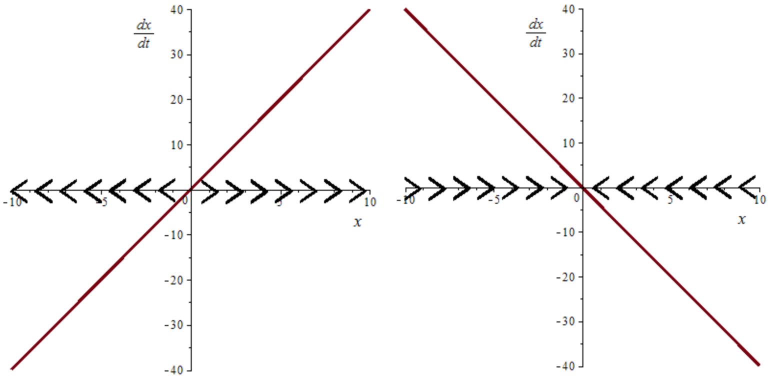
3.1 Linear Equations

Consider the equation $\frac{dx}{dt} = ax$ where a is a non-zero constant.

The solution is $x = x_0e^{at}$. The equilibrium occurs at when $\frac{dx}{dt} = 0$ that is when $ax = 0$ this happens when $x = 0$. Whether the equilibrium is stable or unstable depends on whether $\frac{\partial}{\partial x}(ax)$ is positive or negative. In this case that means it depends on whether the constant a is positive or negative. Therefore there are two cases for the stability, when $a < 0$ and when $a > 0$.

If $a > 0$ then $x \to \pm\infty$ when $t \to 0$. For $a > 0$ $\frac{dx}{dt}$ increases as ax increases and $\frac{dx}{dt}$ decreases as ax decreases, this means that the equilibrium point is unstable.

If $a < 0$ then $x \to 0$ when $t \to \infty$. For $a < 0$ $\frac{dx}{dt}$ decreases as ax increases and $\frac{dx}{dt}$ increases as ax decreases, this means that the equilibrium point is stable.



(a) a positive

(b) a negative

Figure 3.1: Plot of rate of change for linear differential equation $\frac{dx}{dt} = ax$. Arrows on the x-axis give direction of evolution of solution.

3.2 Non-linear Equations

With linear ordinary differential equations it is always possible to find an analytic solution. However with non-linear equations there can be many possible solutions, this results in more than one equilibrium. The simplest non-linear equation is quadratic.

3.2.1 Quadratic right hand side

Consider the equation $\frac{dx}{dt} = f(x) = ax(x-1)$, where a is a non-zero constant.

The equilibrium occurs when $\frac{dx}{dt} = 0$ this happens when either $x = 0$ or $x = 1$. Therefore this equation has 2 equilibrium points $x = 0$ and $x = 1$.

To establish the stability of the equilibrium points we need to look at the derivative of the right hand side at the equilibrium point. When the derivative of the right hand side is negative the equilibrium point is stable. When the derivative is positive the equilibrium point is unstable.

Here $f'(x) = a(x-1) + ax$. Again, like the linear system, the stability depends on the constant a.

When $a > 0$;

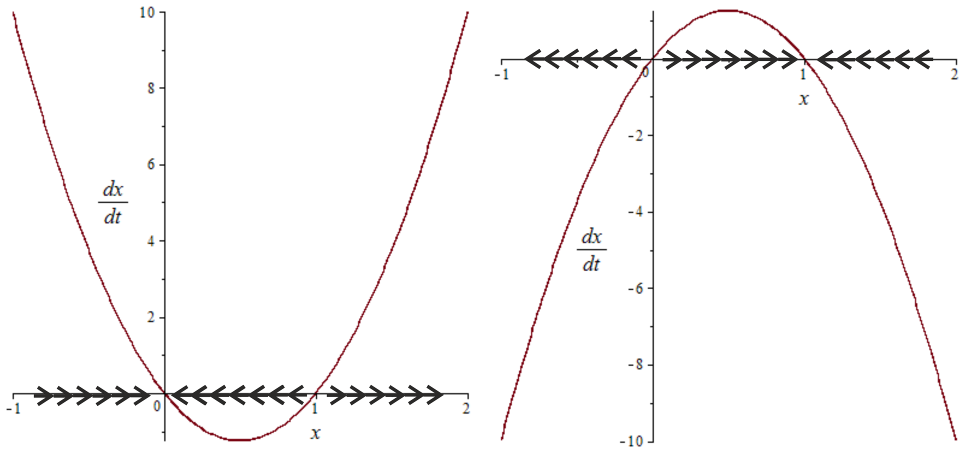
$x = 0$ $f'(x) = -a < 0$ therefore the equilibrium point is stable.

$x = 1$ $f'(x) = a > 0$ therefore the equilibrium point is unstable.

When $a < 0$;

$x = 0$ $f'(x) = -a > 0$ therefore the equilibrium point is unstable.

$x = 1$ $f'(x) = a < 0$ therefore the equilibrium point is stable.



(a) a positive (b) a negative

Figure 3.2: Plot of rate of change for quadratic differential equation $\frac{dx}{dt} = ax(x-1)$. Arrows on the x-axis give direction of evolution of solution.

3.2.2 Cubic right hand side

Consider the equation $\frac{df}{dt} = f(x) = ax(x-1)(x-2)$

The equation has equilibrium at $x = 0$, $x = 1$ and $x = 2$.

$f'(x) = 3ax^2 - 6ax + 2a$. Again the stability of these equilibrium point depends on the sign of the constant a.

When $a > 0$;

$x = 0$ $f'(0) = 2a > 0$ therefore the equilibrium point is unstable.

$x = 1$ $f'(1) = -a < 0$ therefore the equilibrium point is stable.

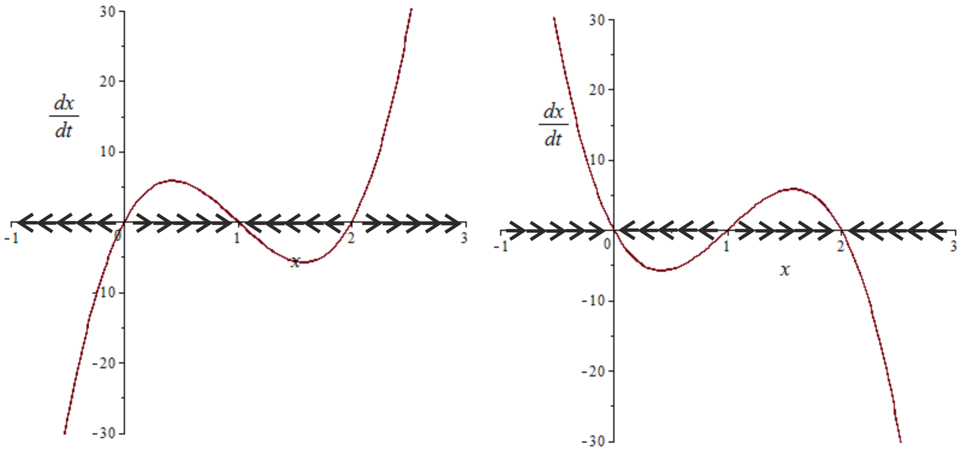
$x = 2$ $f'(2) = 2a > 0$ therefore the equilibrium point is unstable.

When $a < 0$;

$x = 0$ $f'(0) = 2a < 0$ therefore the equilibrium point is stable.

$x = 1$ $f'(1) = -a > 0$ therefore the equilibrium point is unstable.

$x = 2$ $f'(2) = 2a < 0$ therefore the equilibrium point is stable.



(a) a positive (b) a negative

Figure 3.3: Plot of rate of change for cubic differential equation $\frac{df}{dt} = ax(x-1)(x-2)$. Arrows on the x-axis give direction of evolution of solution.

Non-linear differential equations can have many equilibria, the stability of each equilibrium point alternates, and so if you know the stability of one equilibrium point you can work out the stability of all the other points without further calculation, provided all of the equilibriums are non-hyperbolic.

Chapter 4

Fold Bifurcation

Here we will consider non-linear differential equations with a parameter. I will consider how the parameter c affects stability. The value of the c for which the equilibrium is non-hyperbolic, that is $\frac{\partial f}{\partial x}(x, c) = 0$, is called a bifurcation point.

4.1 Example

Consider the equation

$$\frac{dx}{dt} = f(x, c) = x(x-1) + c$$

where c is a parameter.

The equilibrium occurs when $\frac{dx}{dt} = 0$ that is $x^2 - x + c = 0$ this can be solved using the quadratic equation giving;

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4c}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$$

Therefore there are two equilibrium points. The equilibrium points only exist when $c < \frac{1}{4}$ due to the square root.

To analyse the stability of the equilibrium points we need to look at $\frac{\partial f}{\partial x}$ at these points.

$\frac{\partial f}{\partial x}\left(\frac{1}{2} + \sqrt{\frac{1}{4} - c}, c\right) = 2\left(\frac{1}{2} + \sqrt{\frac{1}{4} - c}\right) - 1 = 2\left(\sqrt{\frac{1}{4} - c}\right) > 0.$
Therefore this equilibrium point is unstable and only exists when $c < \frac{1}{4}$.
When $c = \frac{1}{4}$, $\frac{\partial f}{\partial x} = 0$ this means the equilibrium is non-hyperbolic.

$\frac{\partial f}{\partial x}\left(\frac{1}{2}-\sqrt{\frac{1}{4}-c},c\right)=2\left(\frac{1}{2}-\sqrt{\frac{1}{4}-c}\right)-1=-2\left(\sqrt{\frac{1}{4}-c}\right)<0.$
Therefore this equilibrium point is stable and exists when $c<\frac{1}{4}$. When $c=\frac{1}{4}$ the equilibrium is non-hyperbolic.

The equilibrium point is non-hyperbolic at $(x,c)=(\frac{1}{2},\frac{1}{4})$ at this point bifurcation occurs and both equilibrium points disappear. This is fold bifurcation.

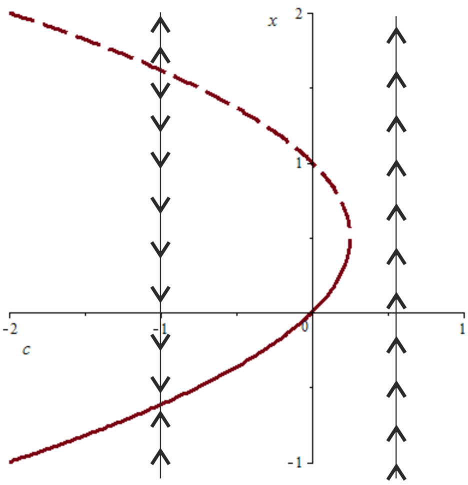


Figure 4.1: Bifurcation Diagram for fold bifurcation on $\frac{dx}{dt}=ax(x-1)+c$. The dashed line represents an unstable equilibrium and the solid line a stable equilibrium. The arrows give direction of evolution of solution.

4.2 Normal form

To find a general way to describe fold bifurcation it is useful to bring equations to the normal form. Once the equations are in normal form there is only four possible options for the orientation and stability. All equations that have fold bifurcation can be transformed into one of these normal forms.

Consider the equation;

$$\frac{dx}{dt}=f(x,c)$$

Assume x^* is an equilibrium value and c^* is a bifurcation value.

This means $f(x^*,c^*)=0$ and $\frac{\partial f}{\partial x}(x^*,c^*)=0$.

To anaylse the equilibrium and bifurcation point we need to analyse the normal form.

If we introduce $\tilde{x} = x - x^*$ and $\tilde{c} = c - c^*$. The equilibrium now occurs at $\tilde{x} = 0$ and the bifurcation now occurs at $\tilde{c} = 0$.

Because the new ODE has equilibrium at $(0, 0)$ we can use the Maclaurin expansion, which is the Taylor series expansion of a function about $(0, 0)$. Because $f'(0, 0) = 0$ we need only expand the function up to second order terms.

The Maclaurin expansion;

$$f(\tilde{x}, \tilde{c}) = f(0,0) + \frac{\partial f}{\partial \tilde{x}}(0,0)\tilde{x} + \frac{\partial f}{\partial \tilde{c}}(0,0)\tilde{c} + \frac{\partial^2 f}{\partial^2 \tilde{x}}(0,0)\frac{\tilde{x}^2}{2} + \frac{\partial^2 f}{\partial \tilde{x}\partial \tilde{c}}(0,0)\tilde{x}\tilde{c} + \frac{\partial^2 f}{\partial^2 \tilde{c}}(0,0)\frac{\tilde{c}^2}{2}$$

Because $f(0, 0) = 0$ and $\frac{\partial f}{\partial \tilde{x}}(0, 0) = 0$ the equation becomes

$$f(\tilde{x}, \tilde{c}) = f_{\tilde{c}}\tilde{c} + f_{\tilde{x}\tilde{x}}\frac{\tilde{x}^2}{2} + f_{\tilde{c}\tilde{c}}\frac{\tilde{c}^2}{2} + f_{\tilde{x}\tilde{c}}\tilde{x}\tilde{c}$$

$$f(\tilde{x}, \tilde{c}) = f_{\tilde{c}}c + \frac{1}{2}f_{\tilde{x}\tilde{x}}\tilde{x}^2 + \frac{1}{2}f_{\tilde{c}\tilde{c}}\tilde{c}^2 + f_{\tilde{x}\tilde{c}}\tilde{x}\tilde{c}$$

Now we can complete the square.

$$f(\tilde{x}, \tilde{c}) = f_{\tilde{c}}\tilde{c} + \frac{1}{2}\left(\sqrt{f_{\tilde{x}\tilde{x}}}\tilde{x} + \frac{f_{\tilde{x}\tilde{c}}}{\sqrt{f_{\tilde{x}\tilde{x}}}}\tilde{c}\right)^2 + \left(\frac{1}{2}f_{\tilde{c}\tilde{c}} - \frac{f_{\tilde{x}\tilde{c}}^2}{f_{\tilde{x}\tilde{x}}}\right)\tilde{c}^2$$

$\tilde{c}$ is small therefore $\tilde{c}^2$ is negligible and so can be ignored. This allows me to remove the third term. I could also remove everything involving $\tilde{c}^2$ from the second term however keeping them will not affect the normal form but will slightly improve accuracy and so I shall keep the second term as it is. This leaves us with;

$$f(\tilde{x}, \tilde{c}) = f_{\tilde{c}}\tilde{c} + \frac{1}{2}\left(\sqrt{f_{\tilde{x}\tilde{x}}}\tilde{x} + \frac{f_{\tilde{x}\tilde{c}}}{\sqrt{f_{\tilde{x}\tilde{x}}}}\tilde{c}\right)^2$$

Let $c = f_c c$ and $x = \frac{1}{\sqrt{2}}\left(\sqrt{f_{\tilde{x}\tilde{x}}}\tilde{x} + \frac{f_{\tilde{x}\tilde{c}}}{\sqrt{f_{\tilde{x}\tilde{x}}}}\tilde{c}\right)$.

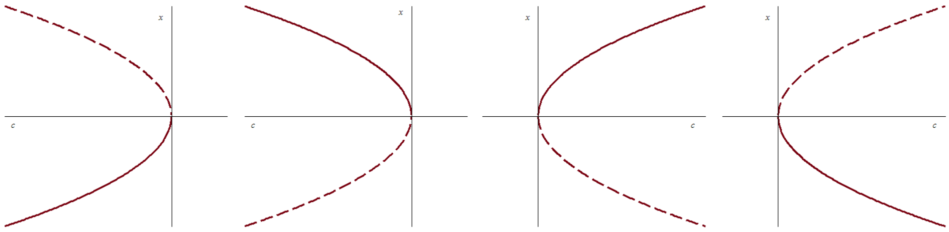
This gives the normal form of fold bifurcation;

$$\frac{dx}{dt} = \pm\, c \pm x^2$$

The sign of c is the same as the sign of $\frac{\partial f}{\partial c}$.

The sign of x^2 is the same as $\frac{\partial^2 f}{\partial x^2}$.

For bifurcation to occur the equilibrium must be non-hyperbolic, that is, $\frac{\partial f}{\partial x} = 0$. For fold bifurcation to occur the following two properties must hold $\frac{\partial f}{\partial c} \neq 0$ and $\frac{\partial^2 f}{\partial x^2} \neq 0$.



(a) $x' = c + x^2$ (b) $x' = -c - x^2$ (c) $x' = c - x^2$ (d) $x' = -c + x^2$

Figure 4.2: Fold bifurcation. Normal form. Four plots correspond to 4 different cases

From the normal form diagrams it can be seen that whether c and x^2 are positive or negative effect the type of fold bifurcation. If c and x^2 are the same sign then the equilibriums vanishe at $c = 0$ and if they are different signs the equilibriums appear at $c = 0$. If x^2 is negative the equilibriums are stable when $x > 0$ and unstable when $x < 0$. If x^2 is positive the equilibriums are stable when $x < 0$ and unstable when $x > 0$.

Chapter 5

Transcritical bifurcation

5.1 Example 1

Consider the equation

$$\frac{dx}{dt} = f(x,c) = xg(x,c) = x(x-1+c)$$

The equation has equilibria at $x = 0$ and $x = 1 - c$.

$\frac{\partial f}{\partial x} = 2x - 1 + c$

Equilibrium at $x = 0$.

$$\frac{\partial f}{\partial x}(0,c) = -1 + c$$

Therefore the equilibrium is stable when $c < 1$ and unstable when $c > 1$ and non-hyperbolic when $c = 1$.

Equilibrium at $x = 1 - c$.

$$\frac{\partial f}{\partial x}(1-c,c) = 1 - c$$

Therefore the equilibrium is stable when $c > 1$ and unstable when $c < 1$ and non-hyperbolic when $c = 1$.

This bifurcation point cannot be a fold bifurcation point because $x = 0$ is always an equilibrium point regardless of the parameter c and so exists before and after the bifurcation point. In fold bifurcation we have a change from zero to two equilibrium depending on the value of the parameter c. The bifurcation point here is a transcritical bifurcation point. In transcritical bifurcation two equilibriums collide and their stability is exchanged.

Transcritical bifurcation occurs at the fixed point $(x, c) = (0, 1)$, at this point the equilibriums collide and their stability is exchanged.

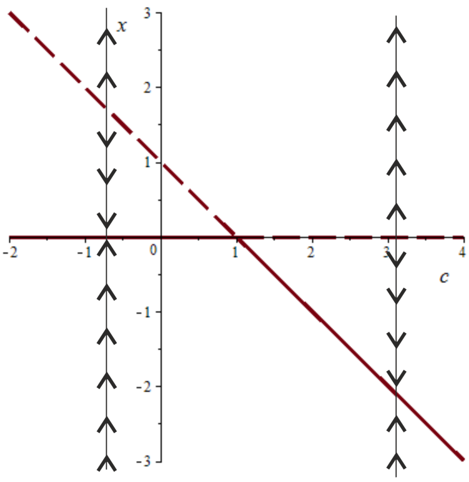


Figure 5.1: Bifurcation Diagram for transcritical bifurcation on $\frac{dx}{dt} = x(x - 1 + c)$. The dashed line represents an unstable equilibrium and the solid line a stable equilibrium. The arrows give direction of evolution of solution.

5.2 Example 2

Consider the equation

$$\frac{dx}{dt} = f(x, c) = x(ax(x-1) + c)$$

The equation has one equilibrium at $x = 0$ and another equilibrium at $x = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, the same as the fold bifurcation example, which is non-hyperbolic when $x = \frac{1}{2}, c = \frac{a}{4}$.

The partial derivative is $\frac{\partial f}{\partial x} = 3ax^2 - 2ax + c$

$\frac{\partial f}{\partial x}(0,0) = c$ and so the equilibrium is stable when $c < 0$ and unstable when $c > 0$ and non-hyperbolic when $c = 0$. Therefore $(0,0)$ is a bifurcation point, because the equilibrium $x = 0$ exists before and after the bifurcation point this is a transcritcal bifurcation.

At $x = \frac{1}{2}, c = \frac{1}{4}$ fold bifurcation occurs, we have the same stability as the fold bifurcation example, that is unstable when $x > \frac{1}{2}$ and stable when $x < \frac{1}{2}$.

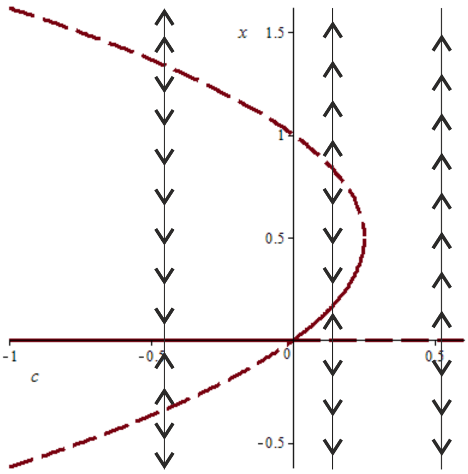


Figure 5.2: Bifurcation diagram for the coexisting transcritical and fold bifurcation simultaneously occurring on $\frac{dx}{dt} = x(ax(x-1)+c)$. The dashed line represents an unstable equilibrium and the solid line a stable equilibrium. The arrows give direction of evolution of solution.

For the equilibrium $x = 0$ you can see that the equilibrium is stable when $c < 0$ and unstable when $c > 0$.

The equilibrium $x = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ is unstable for $c < \frac{1}{4}$ and vanishes at the fold bifurcation point $c = \frac{1}{4}$.

The equilibrium $\frac{1}{2} - \sqrt{\frac{1}{4} - c}$ is unstable when $c < 0$ at $c = 0$ transcritical bifurcation occurs and so the equilibrium is stable for $0 < c < \frac{1}{4}$. At $c = \frac{1}{4}$ fold bifurcation occurs and the equilibrium vanishes.

5.3 Normal form

To make standard analysis we need to bring the equation to the normal form.

Consider the ODE

$$\frac{dx}{dt} = xg(x,c)$$

with non-hyperbolic equilibrium at (x^*, c^*). That is $g(x^*, c^*) = 0$.

In order to use the Taylor expansion I will introduce the change of variables;

$$\tilde{x} = x - x^*, \quad \tilde{c} = c - c^*$$

I can now use the Taylor expansion, in this case we need only the first order terms.

$$f(\tilde{x},\tilde{c}) = x\left(g(0,0) + \frac{\partial g}{\partial \tilde{x}}(0,0)\tilde{x} + \frac{\partial g}{\partial \tilde{c}}(0,0)\tilde{c}\right)$$

Remember $g(0,0) = 0$ and so the expansion becomes;

$$f(\tilde{x},\tilde{c}) = \tilde{x}\left(\frac{\partial g}{\partial \tilde{x}}(0,0)\tilde{x} + \frac{\partial g}{\partial \tilde{c}}(0,0)\tilde{c}\right)$$

$$f(\tilde{x},\tilde{c}) = \tilde{x}(g_{\tilde{x}}\tilde{x} + g_{\tilde{c}}\tilde{c})$$

$$f(\tilde{x},\tilde{c}) = g_{\tilde{x}}\tilde{x}^2 + g_{\tilde{c}}\tilde{c}\tilde{x}$$

$$f(\tilde{x},\tilde{c}) = \sqrt{g_{\tilde{x}}}\tilde{x}\left(\sqrt{g_{\tilde{x}}}\tilde{x} + \frac{g_{\tilde{c}}}{\sqrt{g_{\tilde{x}}}}\tilde{c}\right)$$

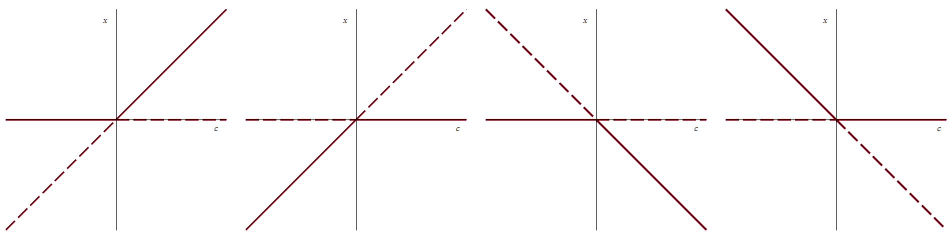
Let $x = \sqrt{g_{\tilde{x}}}\tilde{x}$ and $c = \frac{g_{\tilde{c}}}{\sqrt{g_{\tilde{x}}}}\tilde{c}$

The normal form is therefore

$$\frac{dx}{dt} = x(\pm x \pm c)$$

In transcritical bifurcation there are two equilibriums and one bifurcation point. One equilibrium point exists regardless of parameter c and so the equilibrium point exists before and after the bifurcation point. The stability of the two equilibrium points changes when the equilibria collide.

The conditions for transcritical bifurcation to occur are $\frac{\partial f}{\partial c} = 0$ and $\frac{\partial^2 f}{\partial x^2} \neq 0$.



(a) $x' = x(c - x)$ (b) $x' = x(x - c)$ (c) $x' = x(x + c)$ (d) $x' = x(-x - c)$

Figure 5.3: Transcritical bifurcation normal forms

Chapter 6

Pitchfork Bifurcation

Pitchfork bifurcation occurs when the function $f(x, c)$ is odd, that is when $f(-x, c) = -f(x, c)$, and when there is an equilibrium point at $x = 0$.

6.1 Example

Consider the equation

$$\frac{dx}{dt} = f(x, c) = xc - 4x^3$$

First confirm that $f(x, c)$ is odd.

$$f(-x, c) = -xc + 4x^3 = -f(x, c)$$

$f(x, c)$ has equilibrium at when $f(x, c) = 0$ that is when $xc - 4x^3 = 0$ $x(c - 4x^2) = 0$ this happens when $x = 0$ and $x = \pm\frac{\sqrt{c}}{2}$.

The derivative of the right hand side is $\frac{\partial f}{\partial x} = c - 12x^2$.

When $x = 0$;

$$\frac{\partial f}{\partial x}(0, c) = c - 12(0)^2 = c$$

Therefore the equilibrium at $x = 0$ is stable when $c < 0$, unstable when $c > 0$ and non-hyperbolic when $c = 0$.

The equilibriums at $x = \frac{\pm\sqrt{c}}{2}$ can only occur when $c > 0$ because of the $\sqrt{c}$.

$$\frac{\partial f}{\partial x}\left(\frac{\sqrt{c}}{2}, c\right) = c - 12\left(\frac{\pm\sqrt{c}}{2}\right)^2 = -2c$$

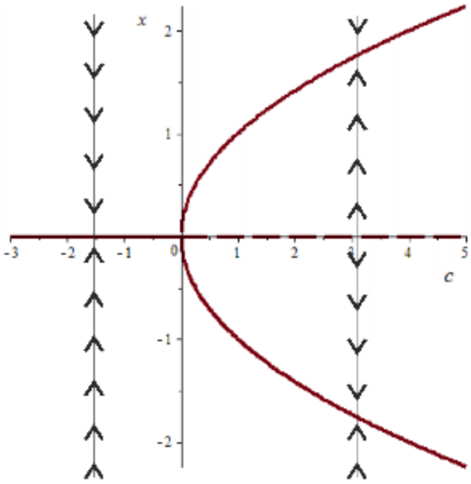


Figure 6.1: Bifurcation diagram for pitchfork bifurcation on $\frac{dx}{dt} = xc - 4x^3$ with stability shown on the cross section

And so the equilibrium point at $x = \pm\frac{\sqrt{c}}{2}$ is always stable.
This is an example of supercritical pitchfork bifurcation.

6.2 Normal form

To make standard analysis we need to bring the equation to the normal form.

Consider the equation

$$\frac{dx}{dt} = f(x,c)$$

Where $f(x,c)$ is an odd function, that is $f(-x,c) = -f(x,c)$.

Assume non-hyperbolic equilibrium at $x = x^*$, $c = c^*$.

Therefore $f(x^*,c^*) = 0$ and $\frac{\partial f}{\partial x}(x^*,c^*) = 0$.

Let $\tilde{x} = x - x^*$, $\tilde{c} = c - c^*$. There is now non-hyperbolic equilibrium at $(0,0)$.

I can now use the Taylor expansion;

$$\begin{aligned} f(\tilde{x},\tilde{c}) &= f(0,0) + \frac{\partial f}{\partial \tilde{x}}(0,0)\tilde{x} + \frac{\partial f}{\partial \tilde{c}}(0,0)\tilde{c} + \frac{\partial^2 f}{\partial \tilde{x}^2}(0,0)\frac{\tilde{x}^2}{2} + \frac{\partial^2 f}{\partial \tilde{x}\partial \tilde{c}}(0,0)\tilde{x}\tilde{c} \\ &+ \frac{\partial^2 f}{\partial \tilde{c}^2}(0,0)\frac{\tilde{c}^2}{2} + \frac{\partial^3 f}{\partial \tilde{x}^3}(0,0)\frac{\tilde{x}^3}{6} + \frac{\partial^3 f}{\partial^2 \tilde{x}\partial \tilde{c}}(0,0)\frac{\tilde{x}^2\tilde{c}}{2} + \frac{\partial^3 f}{\partial \tilde{x}\partial^2 \tilde{c}}(0,0)\frac{\tilde{x}\tilde{c}^2}{2} \\ &+ \frac{\partial^3 f}{\partial^3 \tilde{c}}(0,0)\frac{\tilde{c}^3}{6} \end{aligned}$$

$f(0,0) = 0$ due to the equilibrium at $(0,0)$.

And because the equilibrium is non-hyperbolic, a necessary condition for all bifurcation, $\frac{\partial f}{\partial \tilde{x}}(0,0) = 0$.

Because f is an odd function, all terms that involve $\tilde{x}$ to even powers are zero, and derivatives that involve even powers of $\tilde{x}$ are zero.

If $\frac{\partial f}{\partial \tilde{c}} \neq 0$ even though the equilibrium is non-hyperbolic bifurcation does not occur. All terms involving $\tilde{c}^2$ are small and so can be neglected. I shall only neglect the terms that involve $\tilde{c}^2$ only.

Therefore the expansion becomes;

$$f(\tilde{x}, \tilde{c}) = \tilde{x} \left(\frac{\partial^2 f}{\partial \tilde{x} \partial \tilde{c}} \tilde{c} + \frac{\partial^3 f}{\partial \tilde{x} \partial \tilde{c}^2} \frac{\tilde{c}^2}{2} \right) + \frac{\partial^3 f}{\partial \tilde{x}^3} \frac{\tilde{x}^3}{6}$$

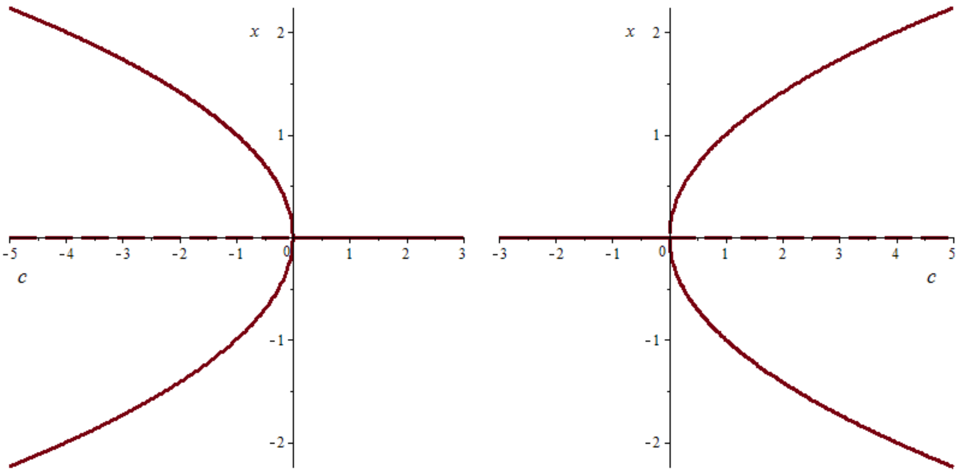
Let $x = \sqrt[3]{\frac{1}{6} \frac{\partial^3 f}{\partial \tilde{x}^3}} \tilde{x}$ and $c = \frac{\frac{\partial^2 f}{\partial \tilde{x} \partial \tilde{c}} \tilde{c} + \frac{\partial^3 f}{\partial \tilde{x} \partial \tilde{c}^2} \frac{\tilde{c}^2}{2}}{\sqrt[3]{\frac{1}{6} \frac{\partial^3 f}{\partial \tilde{x}^3}}}$.

The normal form of pitchfork bifurcation is therefore

$$\frac{dx}{dt} = \pm cx \pm x^3$$

Pitchfork Bifurcation has two types of normal forms; supercritical and subcritical. Whether the bifurcation is supercritical and subcritical depends on the sign of x^3 in the normal form.

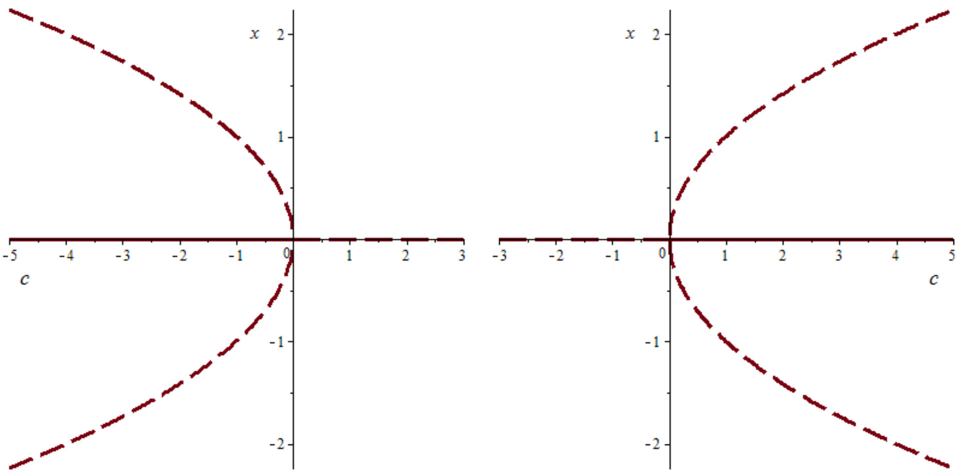
$\frac{dx}{dt} = \pm cx - x^3$ is the supercritical pitchfork normal form.



(a) $x' = x(-c - x^2)$ (b) $x' = x(c - x^2)$

Figure 6.2: Supercritical pitchfork normal form

$\frac{dx}{dt} = \pm cx + x^3$ is the subcritical pitchfork normal form.



(a) $x' = x(c + x^2)$ (b) $x' = x(-c + x^2)$

Figure 6.3: Subcritical pitchfork normal form

For pitchfork bifurcation to occur we must have $\frac{\partial f}{\partial c} = 0$, $\frac{\partial^2 f}{\partial c^2} = 0$, $\frac{\partial^2 f}{\partial x \partial c} \neq 0$, $\frac{\partial^3 f}{\partial x \partial c^2} \neq 0$ and $\frac{\partial^3 f}{\partial x^3} \neq 0$.

In pitchfork bifurcation the ODE either goes from having three equilibrium points before bifurcation to one after the pitchfork bifurcation point. Alternatively it goes from having one equilibrium point before bifurcation to three equilibrium points after pitchfork bifurcation.

Chapter 7

Systems of Ordinary Differential Equations

7.1 System of Linear Equations

7.1.1 General solution

Consider the equations;

$$\begin{cases} \frac{du}{dt} = a_{11}u + a_{12}v \\ \frac{dv}{dt} = a_{21}u + a_{22}v \end{cases}$$

where a_{11}, a_{12}, a_{21} and a_{22} are all positive constants.

The system of linear equations can be rewritten as

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{du}{dt} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

By using the Jacobian of the linear system, here

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and the characteristic equation;

$$\det(J - \lambda I) = 0$$

the eigenvalues can be calculated.

In this example the characteristic equation would be;

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0.$$

And so the eigenvalues can be calculated by solving for λ;

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=0$$

$$\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0$$

$$\lambda_1,\lambda_2=\frac{(a_{11}+a_{22})\pm\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2}$$

Notice that $a_{11}+a_{22}=trace(J)$ and $a_{11}a_{22}-a_{12}a_{21}=det(J)$ and so the equation becomes;

$$\lambda_1,\lambda_2=\frac{TrJ\pm\sqrt{(TrJ)^2-4(detJ)}}{2}$$

If the eigenvalues are real and distinct the solution of the equation is

$$x=c_1\underline{e}_1e^{\lambda_1 t}+c_2\underline{e}_2e^{\lambda_2 t}$$

Where $\underline{e}_{1,2}$ are the eigenvectors corresponding to eigenvalues $\lambda_{1,2}$ and c_1,c_2 are arbitrary constants.

If the eigenvalues are repeated, that is $\lambda_1=\lambda_2=\lambda$, the solution of the equation is

$$x=(c_1+c_2t)\underline{e}e^{\lambda t}$$

Where $\underline{e}$ is the eigenvector corresponding to eigenvalue λ and c_1,c_2 are arbitrary constants.

If the eigenvalues are complex, that is $\lambda=\alpha\pm\beta i$, the solution of the equation is

$$x=e^{\alpha t}(c_1\underline{e}_1\cos\beta t+c_2\underline{e}_2\sin\beta t)$$

Where $\underline{e}_1$ is the eigenvector corresponding to $\lambda=\alpha+\beta i$ and $\underline{e}_2$ is the eigenvector corresponding to $\lambda=\alpha-\beta i$ and c_1,c_2 are arbitrary constants.

The constants in each of the solutions are given by the initial conditions.

$\underline{e}_1$ is an eigenvector of matrix A if $A\underline{e}_1=\lambda_1\underline{e}_1$. If this equation is satisfied λ_1 is an eigenvalue and $\underline{e}_1$ is the eigenvector.

7.1.2 Types of stationary points

Again consider the equations;

$$\begin{cases}\frac{du}{dt}=a_{11}u+a_{12}v\\ \frac{dv}{dt}=a_{21}u+a_{22}v\end{cases}$$

where a_{11}, a_{12}, a_{21} and a_{22} are all positive constants.

At the equilibrium points both variables u and v do not change with respect to time.

Equilibrium can be classified by considering the eigenvalues of the linear system. The number of eigenvalues is equal to the number of state variables, in this case there will be two eigenvalues.

If none of the eigenvalues have zero real part the equilibrium is hyperbolic.

If the eigenvalues are real ($TrJ > 4detJ$) **and negative:** The equilibrium is a sink node which is a stable node. A stable node means that if there is a random pertubation at the equilibrium point the system always returns back to the equilibrium point after the disturbance.

If all of the eigenvalues are real ($TrJ > 4detJ$) **and positive:** The equilibrium is a source node which is an unstable node. An unstable node is where after a small disturbance the system moves away from the equilibrium point.

If all of the eigenvalues are real ($TrJ > 4detJ$) **and one is positive and one is negative** ($\sqrt{(TrJ)^2 - 4detJ} > TrJ$)**:** The equilibrium is a saddle point. A saddle point is always unstable.

If the eigenvalues are complex cunjugate ($TrJ > 4detJ$) **with negative real part** ($TrJ < 0$)**:** The focus is a spiral sink which is stable.

If the eigenvalues are complex cunjugate ($TrJ > 4detJ$) **with positive real part** ($TrJ > 0$)**:** The focus is a spiral source which is unstable.

When at least one eigenvalue has non-zero real part the equilibrium is non-hyperbolic. When both eigenvalues are purely imaginary the equilibrium point is a centre, that is when $TrJ = 0$ the equilibrium is a centre.

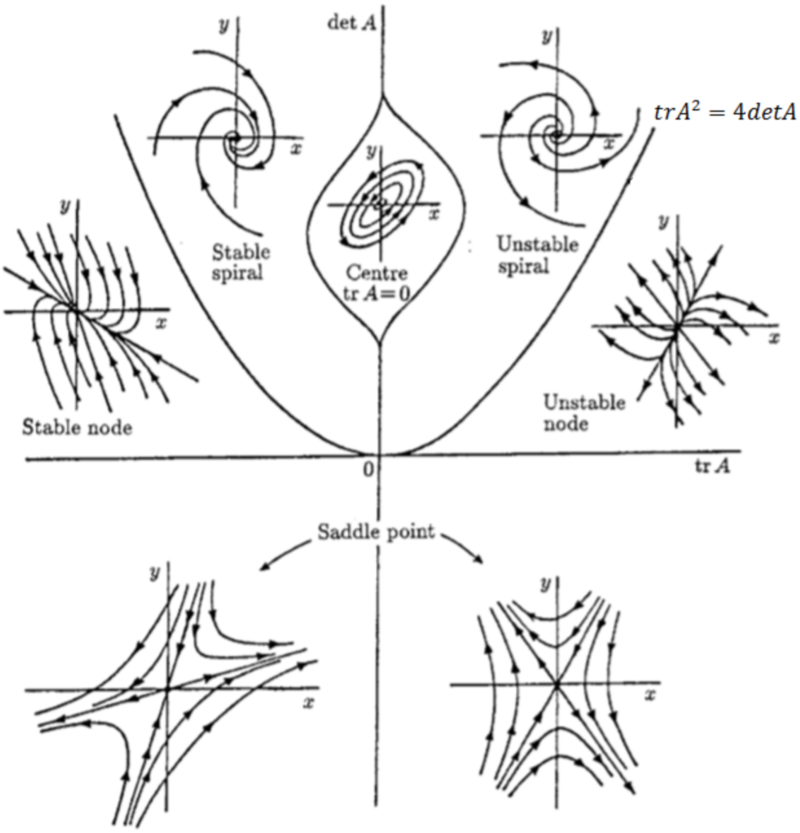


Figure 7.1: Diagram showing how the trace and determinant of J affects the type of equilibrium point

7.1.3 Example 1

Consider the equations

$$\begin{cases} \frac{du}{dt} = 8u - 2v \\ \frac{dv}{dt} = 4u + 2v \end{cases}$$

The characteristic equation for this is;

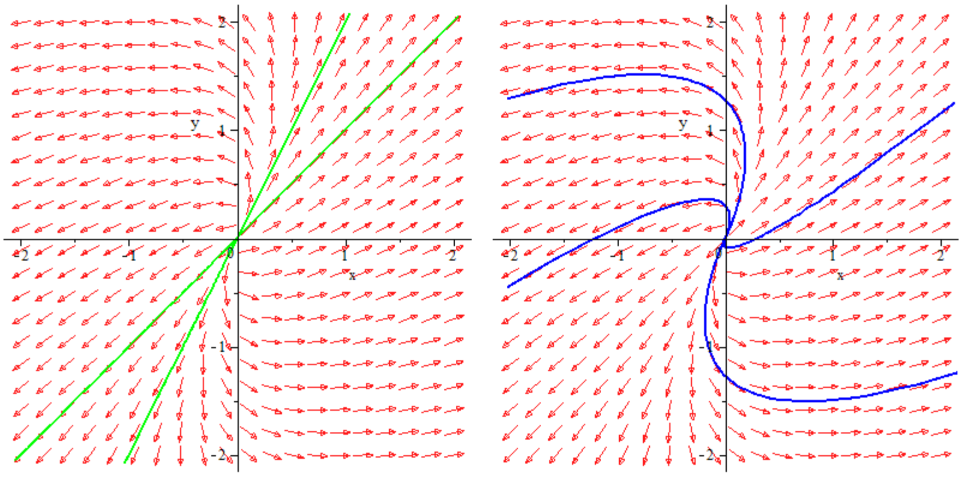
$$\det \begin{pmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix} = 0$$

$$(8-\lambda)(2-\lambda) + 8 = 0 \Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

The eigenvalues are therefore $\lambda_1 = 6$, $\lambda_2 = 4$.

The eigenvalues are real and distinct and so the solution is of the form;

$$x = c_1 \underline{e}_1 e^{\lambda_1 t} + c_2 \underline{e}_2 e^{\lambda_2 t}$$



(a) Direction field with eigenvectors (b) Direction field with phase trajectories

Figure 7.2: Phase Portrait for the sink node

Calculating the eigenvectors;

$\lambda = 6$

$$\begin{pmatrix} 8-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x_1 = x_2$. Therefore the eigenvalue we obtain is $\underline{e_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 4$

$$\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $4x_1 = 2x_2$. Therefore the eigenvalue we obtain is $\underline{e_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The solution is therefore

$$x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{4t}$$

The solution has equilibrium at $(0, 0)$ because the eigenvalues are real and positive this equilibrium is a sink node which is unstable.

7.1.4 Example 2

Consider the equations

$$\begin{cases} \frac{du}{dt} = -2u - 2v \\ \frac{dv}{dt} = 2u - 2v \end{cases}$$

The characteristic equation for this is;

$\det \begin{pmatrix} -2-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = 0$

$(-2-\lambda)(-2-\lambda) + 4 = 0$

$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{4^2 - 32}}{2}$

The eigenvalues are therefore $\lambda_1 = -2 + 2i$, $\lambda_2 = -2 - 2i$.

Calculating eigenvectors;

$\lambda = -2 + 2i$

$$\begin{pmatrix} -2-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2-(-2+2i) & -2 \\ 2 & -2-(-2+2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $-ix_1 = x_2$. And so the eigenvector we obtain is $\underline{e_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda = -2 - 2i$

$$\begin{pmatrix} -2-(-2-2i) & -2 \\ 2 & -2-(-2-2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $ix_1 = x_2$. And so the eigenvector we obtain is $\underline{e_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

The solution to this equation is

$$x = e^{-2t}\left(c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} \cos(2t) + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} \sin(2t)\right)$$

At the equilibrium $-2u - 2v = 0$ and $2u - 2v = 0$. Therefore the only

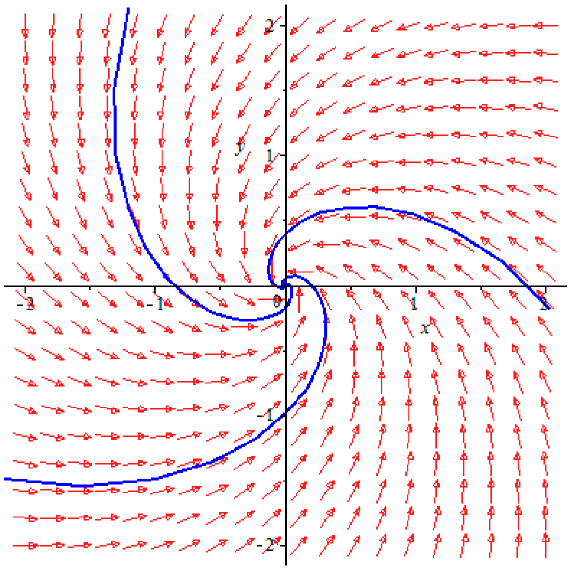


Figure 7.3: Phase portrait for the spiral sink node

equilibrium of this system is $(0, 0)$. Because the eigenvlues are complex cunjugate with negative real part the equilibrium point is a spiral sink which is stable.

7.2 System of Non-Linear Equations

Non-linear systems can rarely be solved exactly, unlike linear systems which can always be solved analytically and therefore exactly. The linearisation technique finds the closest linear system to the non-linear system at the equilibrium, so that the equilibrium can be analysed using the method for a linear system. Non-linear systems can have any number of equilibria.

7.2.1 Linearisation near stationary points

Consider the system of autonomous equations;

$$\begin{cases} \frac{dx}{dt} = F(x, y) \\ \frac{dy}{dt} = G(x, y) \end{cases}$$

with equilibrium at (x^*, y^*). That is;

$$F(x^*, y^*) = 0$$

$$G(x^*, y^*) = 0$$

To linearise, start by approximating $F(x, y)$ and $G(x, y)$ at (x^*, y^*), this is done by calculating the tangent to the functions at the fixed point. This gives;

$$\left\{ \begin{array}{l} \frac{dx}{dt} = F(x,y) \simeq F(x^*,y^*) + \frac{\partial F}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial F}{\partial y}(x^*,y^*)(y-y^*) \\ \frac{dy}{dt} = G(x,y) \simeq G(x^*,y^*) + \frac{\partial G}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial G}{\partial y}(x^*,y^*)(y-y^*) \end{array} \right.$$

Since (x^*, y^*) is a fixed point $F(x^*, y^*) = G(x^*, y^*) = 0$. Thus

$$\left\{ \begin{array}{l} \frac{dx}{dt} = \frac{\partial F}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial F}{\partial y}(x^*,y^*)(y-y^*) \\ \frac{dy}{dt} = \frac{\partial G}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial G}{\partial y}(x^*,y^*)(y-y^*) \end{array} \right.$$

This linear system can be rewritten in matrix notation;

$$\left(\begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \left[\begin{array}{cc} \frac{\partial F}{\partial x}(x^*,y^*) & \frac{\partial F}{\partial y}(x^*,y^*) \\ \frac{\partial G}{\partial x}(x^*,y^*) & \frac{\partial G}{\partial y}(x^*,y^*) \end{array} \right] \left(\begin{array}{c} x-x^* \\ y-y^* \end{array} \right)$$

The Jacobian is

$$\left[\begin{array}{cc} \frac{\partial F}{\partial x}(x^*,y^*) & \frac{\partial F}{\partial y}(x^*,y^*) \\ \frac{\partial G}{\partial x}(x^*,y^*) & \frac{\partial G}{\partial y}(x^*,y^*) \end{array} \right]$$

which can be used to determine the stability of the equilibrium, which because it is now a linear system can be done by calculating the eigenvalues.

The phase portrait of this linear system is close to the phase portrait of the original non-linear system.

7.2.2 Example

Consider the system of equations

$$\left\{ \begin{array}{l} \frac{dx}{dt} = F(x,y) = 2x + 2y \\ \frac{dy}{dt} = G(x,y) = 1 - y^2 \end{array} \right.$$

Equilibrium occurs when $F(x, y) = 0$ and $G(x, y) = 0$. That is;

$2x + 2y = 0 \Rightarrow x = -y$

$1 - y^2 = 0 \Rightarrow y = \pm 1$

Therefore there are equilibria at $(1, -1)$ and $(-1, 1)$

To calculate the Jacobian matrix, we need the partial derivatives;

$\frac{\partial F}{\partial x} = 2$

$\frac{\partial F}{\partial y} = 2$

$\frac{\partial G}{\partial x} = 0$

$\frac{\partial G}{\partial y} = -2y$

The linear approximation in matrix notation is therefore;

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & -2y^* \end{bmatrix} \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}$$

(x^*, y^*)=$(1, -1)$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x - 1 \\ y + 1 \end{pmatrix}$$

The characteristic equation can then be calculated using the Jacobian;

$\begin{vmatrix} 2-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 2$

Therefore the equilibrium at $(1, -1)$ is a source node which is unstable.

(x^*, y^*)=$(-1, 1)$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x + 1 \\ y - 1 \end{pmatrix}$$

The characteristic equation can then be calculated using the Jacobian;

$\begin{vmatrix} 2-\lambda & 2 \\ 0 & -2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-2-\lambda) = 0 \Rightarrow \lambda = \pm 2$

Therefore the equilibrium at $(-1, 1)$ is a saddle point which is unstable.

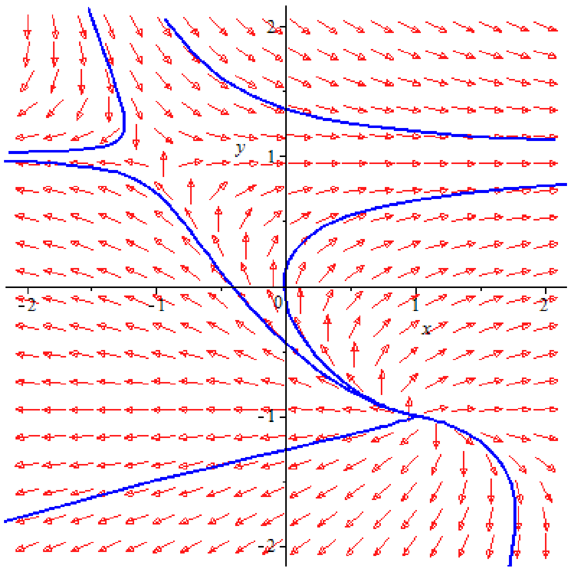


Figure 7.4: Phase portrait for Example 7.2.2 with two equilibrium

From the diagram we can see the saddle point at $(-1, 1)$ and from the direction of the arrows we can see that it is unstable. We can also see the source node at $(1, -1)$ as the arrows show that the flow is going away from the node we can confirm that this is an unstable node.

We can treat each point of the phase plane as an initial condition. Then from the phase portrait, we can see how the solution will change when the initial condition changes.

Chapter 8

Hopf Bifurcation

Hopf bifurcation occurs when the system of non-linear equations is non-hyperbolic. Hopf bifurcation only occurs in systems of two or more dimensions.

In a differential equation Hopf bifurcation typically occurs when the real part of a complex conjugate pair of eigenvalues, of the linearised flow at a fixed point, switch from positive to negative or negative to positive that is when the eigenvalues become purely imaginary.

When the real part of the eigenvalue goes from negative to positive the fixed point goes from being a stable focus to an unstable focus. This bifurcation is called supercritical.

When the real part of the eigenvalue goes form positive to negative the fixed point goes from being an unstable focus to a stable focus. This bifurcation is called subcritical.

8.1 Jordan normal form

Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$.

We can transform matrix A into the canonical form by the transformation $T^{-1}AT$;

$$(\underline{v}_r \, \underline{v}_i)^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\underline{v}_r \, \underline{v}_i) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Where $\underline{v}_r$ is the real part of the eigenvectors of A and $\underline{v}_i$ is the imaginary part of the eigenvectors of A.

Proof:

Let the eigenvalues of matrix A be $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$.

Let $v_{1r} + iv_{1i}$ be the eigenvector corresponding to λ_1.

Let $v_{2r} + iv_{2i}$ be the eigenvector corresponding to λ_2.

Because $v_{1r} + iv_{1i}$ is the complex eigenvector corresponding to λ_1 it satisfies;

$$A\underline{e}_1 = \lambda_1 \underline{e}_1$$

$$A(v_{1r} + iv_{1i}) = (\alpha + i\beta)(v_{1r} + iv_{1i})$$

$$Av_{1r} + Aiv_{1i} = \alpha v_{1r} + \alpha i v_{1i} + i\beta v_{1r} - \beta v_{1i}$$

Equating real and imaginary parts;

Real: $Av_{1r} = \alpha v_{1r} - \beta v_{1i}$

Imaginary: $Av_{1i} = \alpha v_{1i} + \beta v_{1r}$

Because $v_{2r} - iv_{2i}$ is the complex eigenvector corresponding to λ_2 it satisfies;

$$A\underline{e}_2 = \lambda_2 \underline{e}_2$$

$$A(v_{2r} - iv_{2i}) = (\alpha - i\beta)(v_{2r} - iv_{2i})$$

$$Av_{2r} - Aiv_{2i} = \alpha v_{2r} - \alpha i v_{2i} - i\beta v_{2r} - \beta v_{2i}$$

Equating real and imaginary parts;

Real: $Av_{2r} = \alpha v_{2r} - \beta v_{2i}$

Imaginary: $-Av_{2i} = -\alpha v_{2i} - \beta v_{2r} \Rightarrow Av_{2i} = \alpha v_{2i} + \beta v_{2r}$

Therefore;

$$A\underline{v}_r = \alpha \underline{v}_r - \beta \underline{v}_i$$

$$A\underline{v}_i = \alpha \underline{v}_i + \beta \underline{v}_r$$

Let T be a matrix where the columns of the matrix are $\underline{v}_r$ and $\underline{v}_i$.

$T = (\underline{v}_r, \underline{v}_i) = \begin{pmatrix} v_{1r} & -v_{1i} \\ v_{2r} & v_{2i} \end{pmatrix}$

$AT = A(\underline{v}_r, \underline{v}_i)$

$AT = (A\underline{v}_r, A\underline{v}_i)$

$AT = (\alpha\underline{v}_r - \beta\underline{v}_i, \alpha\underline{v}_i + \beta\underline{v}_r)$

$$AT = \begin{pmatrix} \alpha v_{1r} - \beta v_{1i} & \alpha v_{1i} + \beta v_{1r} \\ \alpha v_{2r} - \beta v_{2i} & \alpha v_{2i} + \beta v_{2r} \end{pmatrix}$$
$$AT = \begin{pmatrix} v_{1r} & v_{1i} \\ v_{2r} & v_{2i} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Let $N = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Showing $AT = TN$ shows that $T^{-1}AT = N$.

8.2 Normal form

Consider the system;

$$\begin{cases} \frac{dx}{dt} = f(x, y, c) \\ \frac{dy}{dt} = g(x, y, c) \end{cases}$$

where c is a parameter.

Without loss of generality we can assume that the system has equilibrium at $(0, 0)$, assume that at this equilibrium after linearisation the eigenvalues of the system are $\lambda_{1,2} = \alpha(c) \pm i\beta(c)$ and that for c close to $c = 0$ the eigenvalues have zero real part.

Therefore we have;

$$f(0, 0, c) = 0 \ , \ g(0, 0, c) = 0$$
$$\lambda_{1,2} = \alpha(c) \pm i\beta(c) \ , \ \alpha(0) = 0$$

To study the system around the point $c = 0$ we can expand the right hand side of the system using the Maclaurin series;

$$\begin{cases} \frac{dx}{dt} = f(0, 0, c) + x\frac{\partial f}{\partial x}(0, 0) + y\frac{\partial f}{\partial y}(0, 0) + \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}(0, 0) + xy\frac{\partial^2 f}{\partial x \partial y}(0, 0) \\ +\frac{y^2}{2}\frac{\partial^2 f}{\partial y^2}(0, 0) + ... \\ \frac{dy}{dt} = g(0, 0, c) + x\frac{\partial g}{\partial x}(0, 0) + y\frac{\partial g}{\partial y}(0, 0) + \frac{x^2}{2}\frac{\partial^2 g}{\partial x^2}(0, 0) + xy\frac{\partial^2 g}{\partial x \partial y}(0, 0) \\ +\frac{y^2}{2}\frac{\partial^2 g}{\partial y^2}(0, 0) + ... \end{cases}$$

We know that $f(0, 0, c) = 0$, $g(0, 0, c) = 0$ due to the equilibrium at $(0, 0)$.

The partial derivatives are not constants but functions of the parameter c. If we denote all terms of second order or higher as f_2 and g_2 the system can be rewritten as;

$$\begin{cases} \frac{dx}{dt} = x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0) + f_2(x,y,c) \\ \frac{dy}{dt} = x\frac{\partial g}{\partial x}(0,0) + y\frac{\partial g}{\partial y}(0,0) + g_2(x,y,c) \end{cases}$$

We can now transform the system into the Jordan normal form;

$$\begin{cases} \frac{du}{dt} = \alpha(c)u - \beta(c)v + F_2(u,v,c) \\ \frac{dv}{dt} = \beta(c)u + \alpha(c)v + G_2(u,v,c) \end{cases}$$

If we let $z = u + iv$ and $\bar{z} = u - iv$

$z + \bar{z} = u + iv + (u - iv) = 2u \Rightarrow u = \frac{z+\bar{z}}{2}$

$z - \bar{z} = u + iv - (u - iv) = 2u \Rightarrow v = \frac{z-\bar{z}}{2i}$

Substituting these values of u and v into the Jordan normal form we obtain;

$$\begin{cases} \frac{d\left(\frac{z+\bar{z}}{2}\right)}{dt} = \alpha\frac{z+\bar{z}}{2} - \beta\frac{z-\bar{z}}{2i} + F_2 \\ \frac{d\left(\frac{z-\bar{z}}{2i}\right)}{dt} = \beta\frac{z+\bar{z}}{2} - \alpha\frac{z-\bar{z}}{2i} + G_2 \end{cases}$$

Multiplying the second equation by i we obtain;

$$\frac{d\left(\frac{z-\bar{z}}{2i}\right)i}{dt} = \beta\frac{(z+\bar{z})i}{2} - \alpha\frac{(z-\bar{z})i}{2i} + iG_2$$
$$\frac{d\left(\frac{z-\bar{z}}{2}\right)}{dt} = \beta\frac{(z+\bar{z})i}{2} - \alpha\frac{z-\bar{z}}{2} + iG_2$$

Adding the first equation obtained in Jordan normal form and the new equation;

$$\frac{d\left(\frac{z+\bar{z}}{2}\right)}{dt} + \frac{d\left(\frac{z-\bar{z}}{2}\right)}{dt} = \alpha\frac{z+\bar{z}}{2} - \beta\frac{z-\bar{z}}{2i} + F_2 + \beta\frac{(z+\bar{z})i}{2} - \alpha\frac{z-\bar{z}}{2} + iG_2$$

$$\frac{d}{dt}\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2}\right) = \alpha\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2}\right) + \beta\left(\frac{(z+\bar{z})i}{2} - \frac{z-\bar{z}}{2i}\right) + F_2 + iG_2$$

$$\frac{d}{dt}(z) = \alpha z + \beta z i + F_2 + iG_2$$

Let $\lambda = \alpha + i\beta$ and $F(z, \bar{z}, c) = F_2 + iG_2$

Substituting this gives;

$$\dot{z} = \lambda z + F(z, \bar{z}, c)$$

We can use the Maclaurin series for the function $F(z, \bar{z}, c)$

$$\begin{aligned} F(z,\bar{z},c) &= \frac{\partial^2 F}{\partial z^2}\frac{z^2}{2} + \frac{\partial^2 F}{\partial z \partial \bar{z}} z\bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2}\frac{\bar{z}^2}{2} + \frac{\partial^3 F}{\partial z^3}\frac{z^3}{6} + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}}\frac{z^2\bar{z}}{2} + \frac{\partial^3 F}{\partial z \partial \bar{z}^2}\frac{z\bar{z}^2}{2} \\ &\quad + \frac{\partial^3 F}{\partial \bar{z}^3}\frac{\bar{z}^3}{6} + ... \end{aligned}$$

Our equation then becomes;

$$\begin{aligned} \dot{z} &= \lambda z + \frac{\partial^2 F}{\partial z^2}\frac{z^2}{2} + \frac{\partial^2 F}{\partial z \partial \bar{z}} z\bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2}\frac{\bar{z}^2}{2} + \frac{\partial^3 F}{\partial z^3}\frac{z^3}{6} + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}}\frac{z^2\bar{z}}{2} \\ &\quad + \frac{\partial^3 F}{\partial z \partial \bar{z}^2}\frac{z\bar{z}^2}{2} + \frac{\partial^3 F}{\partial \bar{z}^3}\frac{\bar{z}^3}{6} + ... \end{aligned}$$

We can now work on removing the non-linear terms.

8.2.1 Removing the quadratic term

If we first focus on the quadratic term z^2 we can let our equation become

$$\dot{z} = \lambda z + Az^2$$

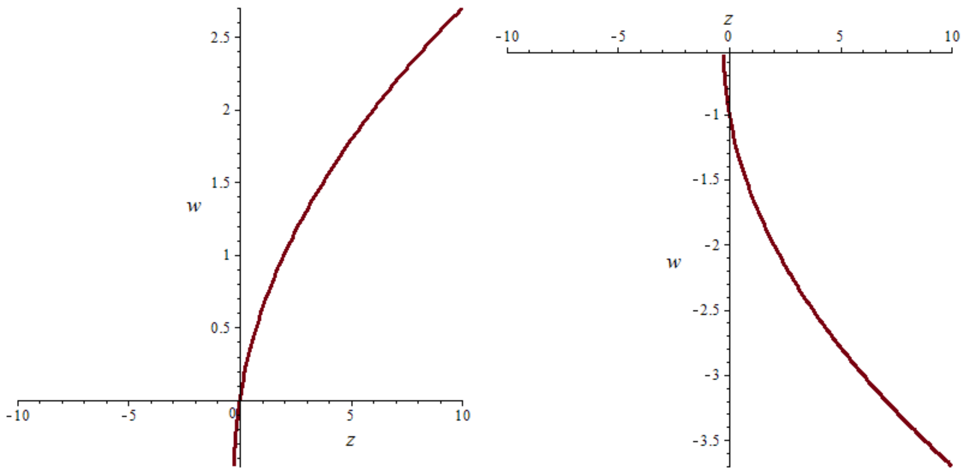
where $A = \frac{1}{2}\frac{\partial^2 F}{\partial z^2}$ which is a function of the parameter c.

Let $z = w + aw^2$

Rearrange to give a quadratic function of w in terms of z

$$aw^2 + w - z = 0 \Rightarrow w = \frac{-1 \pm \sqrt{1 + 4az}}{2a}$$

To see whether we need the plus or minus in the square root we can look at the plot of $w = \frac{-1 \pm \sqrt{1+4az}}{2a}$



(a) $w = \frac{-1+\sqrt{1+4az}}{2a}$ (b) $w = \frac{-1-\sqrt{1+4az}}{2a}$

Figure 8.1: Plots for w

From the plots we can see that we want to take $w = \frac{-1+\sqrt{1+4az}}{2a}$ as this is the plot which passes through $(0,0)$, therefore it maps out equilibrium at $z = 0$ to $w = 0$.

We are interested in our solution near the equilibrium and so we are interested in only small values of z. Because of this we can use the Maclaurin series to further simplify our value of w.

Recall the Maclaurin series for $\sqrt{1+x}$ is $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + ...$

Therefore $\sqrt{1+4az} \approx 1 + 2az - 2a^2z^2 + 4a^3z^3 + ...$

Substituting this equation into our equation for w we obtain;

$$w \approx \frac{-1+1+2az-2a^2z^2+4a^3z^3+...}{2a} = z - az^2 + 2a^2z^3 + ..$$

As we are only aiming to remove the quadratic term now, we can eliminate all terms of order greater than two.
Therefore;

$$w = z - az^2$$

Differentiating gives;

$$\dot{w} = \dot{z} - 2az\dot{z}$$

Recall $\dot{z} = \lambda + Az^2$ and substitute into $\dot{w}$;

$$\dot{w} = \lambda z + Az^2 - 2az(\lambda z + Az^2)$$

$$\dot{w} = \lambda z + (A - 2a\lambda)z^2 - 2aAz^3$$

We are not interested in terms of order greater that two. We shall denote these terms as $O(|z|^3)$

$$\dot{w} = \lambda z + (A - 2a\lambda)z^2 + O(|z|^3)$$

Now if we recall that $z = w + aw^2$ we can substitute this into the equation for $\dot{w}$ and obtain an equation in terms of w, the equation becomes;

$$\dot{w} = \lambda(w + aw^2) + (A - 2a\lambda)(w + aw^2)^2 + O(|z|^3)$$

$$\dot{w} = \lambda w + a\lambda w^2 + Aw^2 + 2Aaw^3 + Aa^2w^4 - 2a\lambda w^2 - 4a^2\lambda w^3 - 2a^3\lambda w^4 + O(|w|^3)$$

$$\dot{w} = \lambda w + a\lambda w^2 + Aw^2 - 2a\lambda w^2 + O(|w|^3)$$

Therefore

$$\dot{w} = \lambda w + (A - a\lambda)w^2$$

To eliminate the quadratic term we need; $A - a\lambda = 0 \Rightarrow a = \frac{A}{\lambda}$

Setting $a = \frac{A}{\lambda}$ we eliminate the quadratic term in $\dot{z} = \lambda z + Az^2$.

$\dot{z}$ will become $\dot{z} = \lambda z + O(|z|^3)$

8.2.2 Removing the complex conjugate quadratic term

Let our equation be

$$\dot{z} = \lambda z + B\bar{z}^2$$

where $B = \frac{1}{2}\frac{\partial^2 F}{\partial \bar{z}^2}$ which is a function of the parameter c.

Let $z = w + b\bar{w}^2$ with complex conjugate $\bar{z} = \bar{w} + \bar{b}w^2$ as $\bar{\bar{w}} = w$

Rewrite to make w and $\bar{w}$ the subject;

$$w = z - b\bar{w}^2$$

$$\bar{w} = \bar{z} - \bar{b}w^2 = \bar{z} - O(|z|^2)$$

As we are only interested in the $\bar{z}^2$ terms we can include all other order two and higher terms in $O(|z|^2)$. Substituting the second equation into the first we obtain;

$$w = z - b(\bar{z} - O(|z|^2))^2$$

$$w = z - b(\bar{z^2} - 2\bar{z}O(|z|^2)) + O(|z|^2)^2$$

$$w = z - b\bar{z^2} + O(|z|^2)$$

And so up to second order terms we have;

$$w = z - b\bar{z}^2$$

If we take the derivative we obtain;

$$\dot{w} = \dot{z} - 2b\bar{z}\dot{\bar{z}}$$

Recall $\dot{z} = \lambda z + B\bar{z}^2$ with complex conjugate $\dot{\bar{z}} = \bar{\lambda}\bar{z} + \bar{B}z^2 = \bar{\lambda}\bar{z} + O(|z|^2)$

Substituting in these values gives;

$$\dot{w} = \lambda z + B\bar{z}^2 - 2B\bar{z}(\bar{\lambda}\bar{z} + O(|z|^2))$$

$$\dot{w} = \lambda z + (B - 2b\bar{\lambda})\bar{z}^2 + O(|z|^2)$$

Recall $z = w + b\bar{w}^2$ and $\bar{z} = \bar{w} + \bar{b}w^2$. Substituting in these values for z and $\bar{z}$ we can obtain an equation for $\dot{w}$ in terms of w and $\bar{w}$ only;

$$\dot{w} = \lambda(w + b\bar{w}^2) + (B - 2b\bar{\lambda})(w + b\bar{w}^2)^2$$

$$\dot{w} = \lambda w + \lambda b\bar{w}^2 + B\bar{w}^2 + 2B\bar{b}w^2 + B\bar{b}^2w^4 - 2b\bar{\lambda}\bar{w}^2 - 4b\bar{b}\bar{\lambda}w^2 - 2b\bar{b}^2\bar{\lambda}w^4$$

$$\dot{w} = \lambda w + (\lambda b + B - 2b\bar{\lambda})\bar{w}^2 + O(|w|^3)$$

To eliminate the quadratic term $\bar{w}^2$ we need $\lambda b + B - 2b\bar{\lambda} = 0 \Rightarrow b = \frac{B}{2\bar{\lambda}-\lambda}$.

Setting $b = \frac{B}{2\bar{\lambda}-\lambda}$ we eliminate the $\bar{w}^2$ quadratic term in $\dot{z} = \lambda z + B\bar{z}^2$.

$\dot{z}$ will become $\dot{z} = \lambda z + O(|z|^3)$

8.2.3 Removing the mixed quadratic term

Let our equation be

$$\dot{z} = \lambda z + Cz\bar{z}$$

with $C = \frac{\partial^2 F}{\partial z \partial \bar{z}}$ which is a function of the parameter c.

Let $z = w + cw\bar{w}$ with complex conjugate $\bar{z} = \bar{w} + \bar{c}w\bar{w}$

Rearranging to make w and $\bar{w}$ the subject;

$$w = z - cw\bar{w}$$

$$\bar{w} = \bar{z} - \bar{c}w\bar{w}$$

Substituting the second equation into the first we obtain;

$$w = z - cw(\bar{z} - \bar{c}w\bar{w})$$

$$z - cw\bar{w} - c\bar{c} - c\bar{c}w^2\bar{w}$$

We are only interested in terms up to order 2;

$$w = z - cw\bar{z} + O(|w|^3)$$

Substituting in the first equation;

$$w = z - c\bar{z}(z - cw\bar{w}) + O(|w|^3)$$

$$w = z - c\bar{z}z - c^2 w\bar{w}z + O(|w|^3)$$

$$w = z - c\bar{z}z + O(|w|^3)$$

Therefore $w = z - c\bar{z}z$ up to second order terms.

Differentiating gives;

$$\dot{w} = \dot{z} - \dot{\bar{z}}z - c\bar{z}\dot{z}$$

Recall $\dot{z} = \lambda z + Cz\bar{z}$ with complex conjugate $\dot{\bar{z}} = \bar{\lambda}\bar{z} + \bar{C}z\bar{z}$

Substituting these values for $\dot{z}$ and $\dot{\bar{z}}$ into $\dot{w}$ gives;

$$\dot{w} = \lambda z + Cz\bar{z} - c(\bar{\lambda}\bar{z} + \bar{C}z\bar{z})z - c\bar{z}(\lambda z + Cz\bar{z})$$

$$\dot{w} = \lambda z + Cz\bar{z} - c\bar{\lambda}\bar{z}z - c\bar{C}z^2\bar{z} - c\lambda\bar{z}z + cCz\bar{z}^2$$

$$\dot{w} = \lambda z + (C - \bar{c}\bar{\lambda} - c\lambda)z\bar{z} + O(|z|^3)$$

Recall $z = w + cw\bar{w}$ and $\bar{z} = \bar{w} + \bar{c}w\bar{w}$

Substituting in these values gives;

$$\dot{w} = \lambda(w + cw\bar{w}) + (C - c\bar{\lambda} - c\lambda)(w + cw\bar{w})(\bar{w} + \bar{c}w\bar{w})$$

$$\dot{w} = \lambda w + c\lambda w\bar{w} + Cw\bar{w} - c\bar{\lambda}w\bar{w} - c\lambda w\bar{w} + O(|w|^3)$$

Therefore up to second order terms;

$$\dot{w} = \lambda w + (C - c\bar{\lambda})w\bar{w}$$

To eliminate $z\bar{z}$ we need $C - c\bar{\lambda} = 0 \Rightarrow c = \frac{C}{\bar{\lambda}}$

Setting $c = \frac{C}{\bar{\lambda}}$ will eliminate $z\bar{z}$ in $\dot{z} = \lambda z + Cz\bar{z}$ and so our equation will become

$$\dot{z} = \lambda z + O(|z|^3)$$

8.2.4 Removing cubic terms

Most of the cubic terms can be removed using similar methods as was used for the quadratic terms. (Taken form Panfilov)

For $\dot{z} = \lambda z + Dz^3$ using the substitution $z = w + dw^3$, $d = \frac{D}{2\lambda}$ is found to be the substitution needed.

For $\dot{z} = \lambda z + Fz\bar{z}^2$ using the substitution $z = w + dw\bar{w}^2$, $f = \frac{F}{2\bar{\lambda}}$ is found to be the substitution needed.

For $\dot{z} = \lambda z + G\bar{z}^3$ using the substitution $z = w + g\bar{w}^3$, $g = \frac{G}{3\bar{\lambda}-\lambda}$ is found to be the substitution needed.

8.2.5 Removing the last cubic term

Let our equation be

$$\dot{z} = \lambda z + Ez^2\bar{z}$$

where $E = \frac{1}{2}\frac{\partial^3 F}{\partial z^2 \partial \bar{z}}$ which is a function of the parameter c.

Let $z = w + ew^2\bar{w}$ with complex conjugate $\bar{z} = \bar{w} + \bar{e}\bar{w}^2 w$

Rearranging to make w and $\bar{w}$ the subject gives;

$$w = z - ew^2\bar{w}$$

$$\bar{w} = \bar{z} - \bar{e}\bar{w}^2 w$$

Substituting in the second equation into the first we obtain;

$$w = z - ew^2(\bar{z} - \bar{e}\bar{w}^2 w)$$

$$w = z - ew^2\bar{z} + e\bar{e}\bar{w}^2 w^3$$

$$w = z - ew^2\bar{z} + O(|w|^5)$$

Now, substituting the value for $\bar{w}$ gives;

$$w = z - e\bar{z}(z - ew^2\bar{w})^2$$

$$w = z - e\bar{z}z^2 + O(|w|^5)$$

Therefore up to third order terms; $w = z - e\bar{z}z^2 + O(|w|^5)$

Taking the derivative gives; $\dot{w} = \dot{z} - e\dot{\bar{z}}z^2 - 2e\bar{z}z\dot{z}$

Recall $\dot{z} = \lambda z + Ez^2\bar{z}$ with complex conjugate $\dot{\bar{z}} = \bar{\lambda}\bar{z} + \bar{E}\bar{z}^2 z$

Substituting these values into $\dot{w}$ gives;

$$\dot{w} = \lambda z + Ez^2\bar{z} - ez^2(\bar{\lambda}\bar{z} + \bar{E}\bar{z}^2 z) - 2e\bar{z}z(\lambda z + Ez^2\bar{z})$$

$$\dot{w} = \lambda z + Ez^2\bar{z} - e\bar{\lambda}z^2\bar{z} - \bar{E}e\bar{z}^2z^3 - 2e\lambda\bar{z}z^2 - 2eEz^3\bar{z}^2$$

$$\dot{w} = \lambda z + (E - e\bar{\lambda} - 2e\lambda)z^2\bar{z} + O(|z|^3)$$

Substituting in $z = w + ew^2\bar{w}$ and $\bar{w} = \bar{w} + \bar{e}\bar{w}^2 w$ we obtain an equation for $\dot{w}$ in terms of w and $\bar{w}$ only;

$$\dot{w} = \lambda(w + ew^2\bar{w}) + (E - e\bar{\lambda} - 2e\lambda)(w + ew^2\bar{w})(\bar{w} + \bar{e}\bar{w}^2 w)$$

$$\dot{w} = \lambda w + e\lambda w^2\bar{w} + (E - e\bar{\lambda} - 2e\lambda)w^2\bar{w}$$

$$\dot{w} = \lambda w + (E - e\bar{\lambda} - e\lambda)w^2\bar{w}$$

$$\dot{w} = \lambda w + (E - e(\lambda + \bar{\lambda}))w^2\bar{w}$$

Recall $\lambda = \alpha(c) + i\beta(c)$ and $\bar{\lambda} = \alpha(c) - i\beta(c)$ and at the bifurcation point, $c = 0$, $\alpha(c) = 0$.

Therefore $\lambda(0) + \bar{\lambda}(0) = i\beta(0) - i\beta(0) = 0$

$\Rightarrow e(\lambda + \bar{\lambda}) = 0$ for all values of e

Therefore $Ew^2\bar{w}$ cannot be removed regardless of the value of e.

And so our equation;

$$\begin{aligned}\dot{z} \quad = \quad & \lambda z + \frac{\partial^2 F}{\partial z^2}\frac{z^2}{2} + \frac{\partial^2 F}{\partial z \partial \bar{z}} z\bar{z} + \frac{\partial^2 F}{\partial \bar{z}^2}\frac{\bar{z}^2}{2} + \frac{\partial^3 F}{\partial z^3}\frac{z^3}{6} + \frac{\partial^3 F}{\partial z^2 \partial \bar{z}}\frac{z^2\bar{z}}{2} \\ & + \frac{\partial^3 F}{\partial z \partial \bar{z}^2}\frac{z\bar{z}^2}{2} + \frac{\partial^3 F}{\partial \bar{z}^3}\frac{\bar{z}^3}{6} + \ldots\end{aligned}$$

Can only be reduced to the form

$$\dot{w} = \lambda w + \delta w^2\bar{w}$$

8.2.6 Polar coordinates

Recall w is a complex number $w = u + iv$

We want to find bifurcation of the phase portrait in the u, v plane. It will be convenient to use polar coordinates.

Lets introduce ρ and ϕ. Let $u = \rho\cos\phi$ and $v = \rho\sin\phi$.

$$w = u + iv = (\rho\cos\phi + i\rho\sin\phi) = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$$

Recall λ and δ are complex too, let $\lambda = \alpha + i\beta$ and $\delta = l + id$

$$\dot{w} = \lambda w + \delta w^2\bar{w} = (\alpha + i\beta)w + (l + id)w^2\bar{w}$$

If $w = \rho e^{i\phi}$;

$$\begin{aligned} \dot{w} &= \dot{\rho}e^{i\phi} + i\dot{\phi}\rho e^{i\phi} \\ w^2 &= (\rho e^{i\phi})(\rho e^{i\phi}) = \rho^2 e^{2i\phi} \\ \bar{w} &= \rho e^{-i\phi} \\ w^2\bar{w} &= (\rho^2 e^{2i\phi})(\rho e^{-i\phi}) = \rho^3 e^i\phi \end{aligned}$$

Substituting into our equation for $\dot{w}$;

$$\dot{\rho}e^{i\phi} + i\dot{\phi}\rho e^{i\phi} = (\alpha + i\beta)(\rho e^{i\phi}) + (l + id)(\rho^3 e^i\phi)$$

Dividing through by $e^{i\phi}$;

$$\dot{\rho} + i\dot{\phi}\rho = (\alpha + i\beta)\rho + (l + id)\rho^3$$

$$\dot{\rho} + i\dot{\phi}\rho = \alpha\rho + i\beta\rho + l\rho^3 + id\rho^3$$

Now we equate the real and imaginary parts;

Real part;

$$\dot{\rho} = \alpha\rho + l\rho^3$$

Imaginary part;

$$\dot{\phi} = \beta + d\rho^2$$

In the second equation if $\beta \neq 0$ then $|\rho| << \beta$ and so we can neglect ρ^2 compared to β. The system then becomes;

$$\begin{cases} \dot{\rho} = \alpha\rho + l\rho^3 \\ \dot{\phi} = \beta \end{cases}$$

Recall α is a function of the parameter c, as the eigenvalues λ are dependent on the parameter c. If $\frac{\partial\alpha}{\partial c} \neq 0$ we can introduce a new parameter μ. μ will have sign - if $\frac{\partial\alpha}{\partial c} < 0$ and sign + if $\frac{\partial\alpha}{\partial c} > 0$. The system becomes;

$$\begin{cases} \dot{\rho} = \pm\mu\rho + l\rho^3 \\ \dot{\phi} = \beta \end{cases}$$

Next rescaling the amplitude, dividing the first equation by l and the second by β we obtain;

$$\begin{cases} \dot{r} = \pm\gamma r \pm r^3 \\ \dot{\theta} = 1 \end{cases}$$

where $\gamma = \frac{\mu}{l}$ and the sign of r^3 comes from the sign of l.

The second equation shows that the angle θ goes around at a constant rate. The first equation gives us the radius which is dependent on the parameter γ.

8.2.7 Cartesian coordinates

To return to Cartesian coordinates from

$$\dot{r} = \pm\gamma r \pm r^3$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

Differentiating gives;

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta$$

First looking at $\dot{x}$. Substitute in the value for $\dot{r}$;

$$\dot{x} = (\pm\gamma r \pm r^3)\cos\theta - r\sin\theta$$

$$\dot{x} = \pm\gamma r\cos\theta \pm r^3\cos\theta - r\sin\theta$$

Recall $x = r\cos\theta$ and $y = r\sin\theta$ and substitute;

$$\dot{x} \pm \gamma x \pm r^3\cos\theta - y$$

If $x = r\cos\theta$ then $x^2 = r^2\cos^2\theta$ and $y = r\sin\theta$ so $y^2 = r^2\sin^2\theta$ therefore $x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$;

$$\dot{x} = \pm\gamma x \pm (x^2 + y^2)r\cos\theta - y$$

$$\dot{x} = \pm\gamma x \pm (x^2 + y^2)x - y$$

$$\dot{x} = -y + x[\pm\gamma \pm (x^2 + y^2)]$$

Using the same substitutions in $\dot{y}$;

$$\dot{y} = (\pm\gamma r \pm r^3)\sin\theta + r\cos\theta$$

$$\dot{y} = \pm\gamma r\sin\theta \pm r^3\sin\theta + r\cos\theta$$

$$\dot{y} = \pm\gamma y \pm (x^2 + y^2)y + x$$

Therefore our system in polar coordinates becomes;

$$\begin{cases} \frac{dx}{dt} = -y + x[\pm\gamma \pm (x^2 + y^2)] \\ \frac{dy}{dt} = x + y[\pm\gamma \pm (x^2 + y^2)] \end{cases}$$

8.3 Limit cycle

A limit cycle is an isolated closed trajectory. Isolated means that the neighbouring trajectories are not closed therefore they spiral either towards or away from the limit cycle.

Limit cycles occur in Hopf Bifurcation. The stability of the limit cycle is determined by the stability of the equilibriums surrounding it. If the limit cycle is stable the neighbouring trajectories approach the limit cycle and if the limit cycle is unstable the neighbouring trajectories go away from the limit cycle.

For the system

$$\frac{dx}{dt} = f(x, y, c)$$
$$\frac{dy}{dt} = g(x, y, c)$$

where c is a parameter.

Assume the system has non-hyperbolic equilibrium at $x = 0$,$y = 0$,$c = 0$. And the eigenvalues of the Jacobian are $\lambda_{1,2} = \alpha(c) \pm i\beta(c)$ and that for c close to $c = 0$ the eigenvalues have zero real part that is $\alpha(0) = 0$.

Hopf Bifurcation takes place if;
$\frac{\partial\alpha}{\partial c}(0) \neq 0$ as this gives $\pm\gamma$
$\beta(0) \neq 0$ otherwise the eigenvalues would be equal to zero at the equilibrium
$Re(c) \neq 0$ as this gives $\pm r^3$

8.4 Normal form diagrams

Consider;

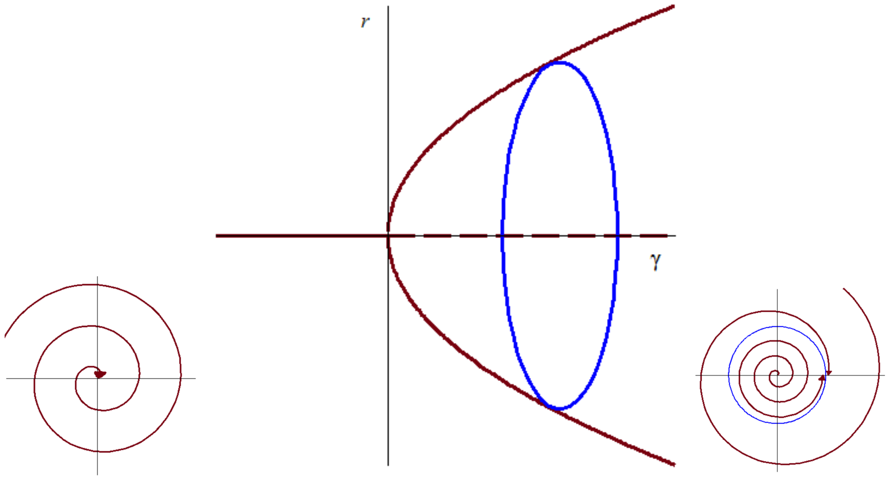
$$\begin{cases} \dot{r} = f(\gamma, r) = \gamma r - r^3 = r(\gamma - r^2) \\ \\ \dot{\theta} = 1 \end{cases}$$

$f(\gamma, r) = 0 \iff r(\gamma - r^2) = 0 \Rightarrow r = 0$ or $r = \pm\sqrt{\gamma}$ which only exists when $\gamma > 0$.

$\frac{\partial f}{\partial r} = \gamma - 3r^2$

When $r = 0$; $\frac{\partial f}{\partial r} = \gamma$ therefore the equilibrium at $r = 0$ is stable when $\gamma < 0$ and unstable when $\gamma > 0$.

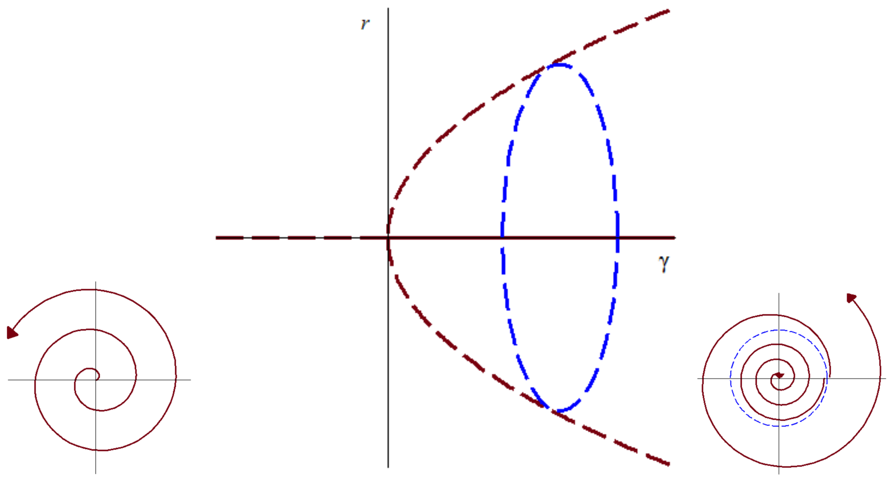
When $r = \pm\sqrt{\gamma}$; $\frac{\partial f}{\partial r} = \gamma - 3(\sqrt{\gamma})^2 = -2\gamma$ because these equilibrium points only exist when $\gamma > 0$ they are always stable.



(a) $\gamma < 0$ (b) Bifurcation with stable limit cycle (c) $\gamma > 0$

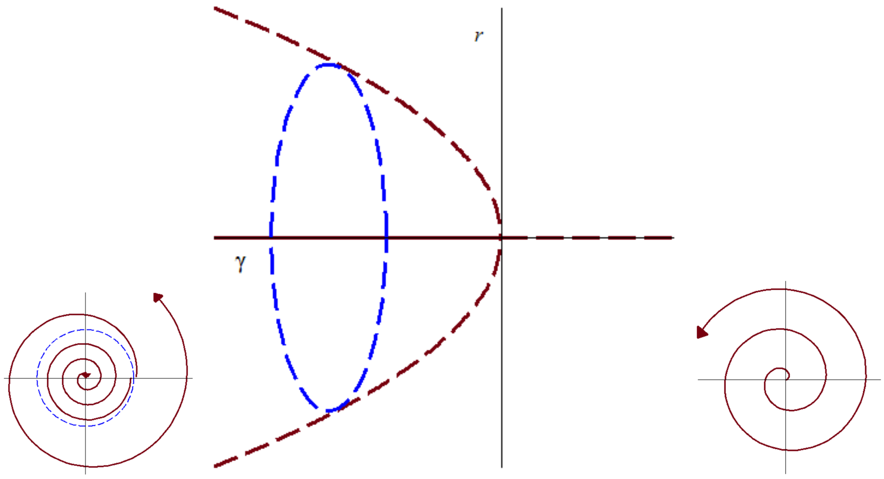
Figure 8.2: Hopf Bifurcation in the system $\dot{r} = \gamma r - r^3$,$\dot{\theta} = 1$

When $\gamma < 0$ there is one stable equilibrium at $r = 0$ and no limit cycle. Bifurcation occurs at $\gamma = 0$. When $\gamma > 0$ there is an unstable equilibrium at $r = 0$, stable equilibriums when $r = \pm\sqrt{\gamma}$ and a stable limit cycle.



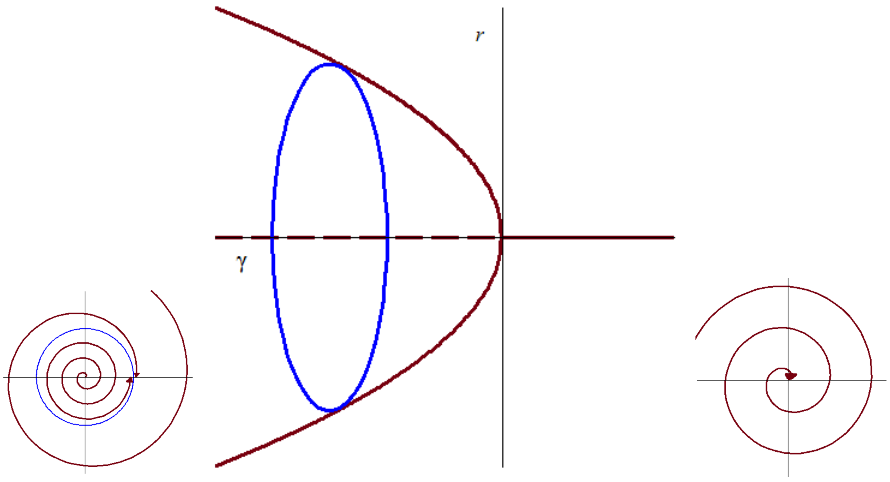
(a) $\gamma < 0$ (b) Bifurcation with unstable limit cycle (c) $\gamma > 0$

Figure 8.3: Hopf Bifurcation $\dot{r} = -\gamma r + r^3$,$\dot{\theta} = 1$



(a) $\gamma < 0$ (b) Bifurcation with unstable limit cycle (c) $\gamma > 0$

Figure 8.4: Hopf Bifurcation $\dot{r} = \gamma r + r^3$,$\dot{\theta} = 1$



(a) $\gamma < 0$ (b) Bifurcation with stable limit cycle (c) $\gamma > 0$

Figure 8.5: Hopf Bifurcation $\dot{r} = -\gamma r - r^3$,$\dot{\theta} = 1$

Chapter 9

Conclusion

There are three possible types of bifurcation in ordinary differential equations; fold, transcritical and pitchfork. For any bifurcation to occur the equilibrium point must be non-hyperbolic, this condition is necessary but not sufficient.

Fold bifurcation requires $\frac{\partial f}{\partial c} \neq 0$ and $\frac{\partial^2 f}{\partial x^2} \neq 0$.

Transcritical bifurcation requires $\frac{\partial f}{\partial c} = 0$ and $\frac{\partial^2 f}{\partial x^2} \neq 0$.

Pitchfork bifurcation requires $\frac{\partial f}{\partial c} = 0$, $\frac{\partial^2 f}{\partial c^2} = 0$, $\frac{\partial^2 f}{\partial x \partial c} \neq 0$, $\frac{\partial^3 f}{\partial x \partial c^2} \neq 0$ and $\frac{\partial^3 f}{\partial x^3} \neq 0$.

In systems of ordinary differential equations Hopf bifurcation can occur if there is a non-hyperbolic equilibrium and the system has complex conjugate eigenvalues $\lambda_{1,2} = \alpha(c) \pm i\beta(c)$ and that for c close to its bifurcation value the eigenvalues have zero real part.

Bifurcation theory is useful in many areas. For example it has biological applications wherein it provides a framework to help to understand the behaviour of networks which have been modelled as dynamical systems. Bifurcation theory also has applications in semi-classical and quantum physics, for example it has been useful in studying laser dynamics.

To continue studying bifurcation it might be interesting to consider bifurcation in continuous systems. It might also be interesting to look at stability in non-linear systems with diffusion for example the conditions for Turing instability.

Chapter 10

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