Stability conditions on triangulated categories

Background:

Triangulated category = additive category $\mathcal{D}$ with
- shift functor $D \xrightarrow{\sim} D$ $E \to E[1]$
- collection of "exact triangles" $E \to F \to G \to E[1]$

satisfying some axioms:

(e.g. for $E \xrightarrow{f} F$ morph in $\mathcal{D}$, $\exists\, Z = \mathrm{cone}(f)$ with

$E \xrightarrow{f} F \to Z \to E[-1]$)

Example: $D^b(A)$ derived category, A abelian category.

Heart of a bounded t-structure: nice abelian category $\mathcal{A}$

in triangulated category $\mathcal{D}$

Def: The heart of a bounded t-structure in $\mathcal{D}$

is full additive subcategory $\mathcal{A} \subset \mathcal{D}$ s.th

(1) For $k_1 > k_2$ $\mathrm{Hom}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$

(2) For every obj. $E \in \mathcal{D}$ $\exists$ integers $k_1 > k_2 \dots > k_n$ and

sequence of exact triangles

$$0 = E_0 \to E_1 \to \dots\dots E^{n-1} \to E^n = E$$
$$\nwarrow A_1 \swarrow \qquad\qquad \nwarrow A_n \swarrow \quad \text{with } A_i \in \mathcal{A}[k_i]$$

Trivial example: $\mathcal{A} \subset D^b(\mathcal{A})$.

(2) is the filtration by cohomology $A_i = H^{-k_i}(E)[k_i]$.

In general the A_i are called the cohomology objects $H^i_{\#}(E)$ of E wrt $\mathcal{A}^{\#}$.

Exercise: If $\mathcal{A}^{\#} \subset D$ is the heart of a bounded t-structure and

$A \to B \to C \to A[1]$ exact triangle with $A, B \in \mathcal{A}^{\#}$

Then $H^i(C) = 0$ unless $i = -1, 0$.

(2) $\mathcal{A}^{\#}$ is an abelian category by defining

$\ker(f: A \to B) := H^{-1}_{\#}(\operatorname{cone} f)$,

$\operatorname{coker}(\ldots) = H^0_{\#}(\operatorname{cone} f)$.

(3) Any exact triangle $A \to B \to C \to A[1]$.

induces long exact sequence of cohomology objects.

$H^i_{\#}(A) \to H^i_{\#}(B) \to \ldots$

Torsion pair and tilt

Defn: A torsion pair on an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories s.t.

(1) $\operatorname{Hom}(\mathcal{T}; \mathcal{F}) = 0$

(2) For all $E \in \mathcal{A}$ $\exists$ exact sequence
$$0 \to T \to E \to F \to 0, \quad T \in \mathcal{T}, \ F \in \mathcal{F}.$$

Example: $\mathcal{A} = \mathrm{Coh}(X)$, $\mathcal{T}$ = torsion sheaves
$\mathcal{F}$ = torsion free sheaves.

Definition/Proposition: Let $\mathcal{T}, \mathcal{F}$ be a torsion pair on $\mathcal{A}$.
Then $\mathcal{A}^{\#} := \{ E \in \mathcal{D}^b(\mathcal{A}) \mid H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, \ H^i(E) = 0 \ \ i \neq 0, -1 \}$
is the heart of a bounded t-structure on $\mathcal{D}^b(\mathcal{A})$
the tilt of $\mathcal{A}$ at $(\mathcal{T}, \mathcal{F})$.

Objects of $\mathcal{A}$ can be thought of extensions $0 \to T \to E \to F \to 0$
" " $\mathcal{A}^{\#}$ - - - - - $0 \to F[-1] \to E \to F \to 0$.

(More precisely $E^{-1} \xrightarrow{d} E^0$, ker $d \in \mathcal{F}$, coker $d \in \mathcal{T}$.

$$\overbrace{\mathcal{T}[0] \ \mathcal{F}[0]}^{\mathcal{A}[0]} \ \overbrace{\mathcal{T}[-1] \ \mathcal{F}[1]}^{\mathcal{A}[-1]}$$

$$\underbrace{\mathcal{T}[0]}_{\mathcal{A}^{\#}[0]} \ \underbrace{\mathcal{F}[0] \ \mathcal{T}[-1]}_{\mathcal{A}^{\#}[-1] \ldots}$$

More generally this works for any triangulated category $\mathcal{D}$, and the heart of a bounded t-structure $\mathcal{A}$.
Make same def for $\mathcal{A}^{\#}$. Replace $H^i(E)$ by cohom obj. wrt t-structure

Example of torsion pair.

C smooth proj. curve, $\mu \in \mathbb{R}$, $\mathcal{A} = \mathrm{Coh}(C)$

$\mathcal{A}_{\geq \mu}$ subcat gen by torsion sheaves, and VB whose HN quotients have slope $\geq \mu$.

$\mathcal{A}_{<\mu}$ vector bundles whose HN quotients have slope $<\mu$.

$\Rightarrow (\mathcal{A}_{\geq \mu}, \mathcal{A}_{<\mu})$ is a torsion pair

(1) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$: If $\mu_{max}(E) > \mu_{min}(E')$

$\Rightarrow \mathrm{Hom}(E, E') = 0$.

(2) Tn HN for E

$0 = \mathcal{F}_0 \subset \ldots \subset \mathcal{F}_n = E$ with $\mathcal{E}_i := \mathcal{F}_i / \mathcal{F}_{i-1}$ semistable of slope μ_i

Take $T = \mathcal{F}_i$, i maximal with $\mu_i \geq \mu$, $F = E/\mathcal{F}_i$.

<u>Remark</u>: One can iterate the process of tilting

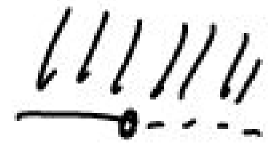
In particular one finds for space of Bridgeland stability conditions, that if one knows the heart for one point in the space, all others are obtained by tilts.

Stability conditions on a triangulated category

Recall stability function on abelian category

$\mathcal{A}$ abelian category, $Z : K(\mathcal{A}) \to \mathbb{C}$ group homom.

s.t. $\forall E \in \mathcal{A}$, $E \neq 0$: $\quad Z(E) \in \mathbb{H} = \{ z = m e^{i\pi\phi} \mid m > 0,\ \phi \in (0,1] \}$



$\mathbb{H}$.

The phase $\phi(E) \in (0,1]$ is $\phi(E) = \frac{1}{\pi} \arg(Z(E))$.

E is called Z-semistable if $\phi(A) \le \phi(E)$ for all subobjects $A \subset E$.

<u>Example</u>: C smooth proj. curve, $\mathcal{A}$ coh. sheaves on C.

$$\mu(E) = \frac{\deg(E)}{rk(E)}\ ; \quad \mu(T) = \infty \quad \text{for } T \text{ torsion}$$

$Z(E) = i\, rk(E) - \deg(E) \in \mathbb{H}$ gives usual slope semistability

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Definition of stability conditions

We make def. in two steps: separated phases and H-N filtration from stability function.

<u>Definition</u>: A <u>slicing</u> $\mathcal{P}$ of a triangulated category $\mathcal{D}$ is a collection of full additive subcategories $\{ \mathcal{P}(\phi) \mid \phi \in \mathbb{R} \}$ s.t.

(1) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

(2) If $\phi_1 > \phi_2 \Rightarrow \mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$

(3) For every $0 \neq E \in \mathcal{D}$ have:

sequence $\phi_1 > \phi_2 > \dots > \phi_n \in \mathbb{R}$ and exact triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E^1 \arrow[r] \arrow[dl] & \dots \arrow[r] & E^{n-1} \arrow[r] & E^n = E \arrow[dl] \\
A^1 \arrow[u, dashed] & & & A^n \arrow[u, dashed] &
\end{tikzcd}$$

With $A_i \in \mathcal{P}(\phi_i)$ (Harder Narasimhan filtration of E)

<u>Remark</u>: (1) Objects of $\mathcal{P}(\phi)$ are called semistable of phase ϕ.

(2) The ϕ_i and the HN filtration are unique.

(3) Put $\phi^+(E) = \phi_1$, $\phi^-(E) = \phi_n$

If $\phi^-(A) > \phi^+(B)$, then $\mathrm{Hom}(A, B) = 0$ (from (2))

(4) If $\mathcal{P}(\phi) = 0$ only for $\phi \in \mathbb{Z}$, then

a slicing is the same as a bounded t-structure with heart $\mathcal{A} = \mathcal{P}(0)$

(5) If $\mathcal{P}$ is a slicing. Set $\mathcal{A} = \mathcal{P}((0,1])$ full subcat of obj. E with $\phi_P^+(E) \leq 1$, $\phi_P^-(E) > 0$.

Then $\mathcal{A}$ is the heart of a bounded t-structure.

i.e. a slicing is a refinement of a bounded t-structure.

Definition: A stability condition on a triangulated category $\mathcal{D}$ is a pair (Z, P) with

$Z : K(\mathcal{D}) \to \mathbb{C}$ group homom. called central charge.

and P a slicing s.th $\forall_{0 \neq E \in P(\phi)}$ we have

$$Z(E) = m(E) \cdot e^{i\pi\phi} \text{ for some } m(E) \in \mathbb{R}_{>0}.$$

Now we want to see that to give a stability condition on $\mathcal{D}$ is the same as giving the heart of a bounded t-structure $\mathcal{A}$ + a stability function on $\mathcal{A}$ compatible with H-N filtration.

And this is how stability conditions are practically constructed.

Proposition: To give a stability cond. (Z, P) on $\mathcal{D}$ is equiv to giving a heart $\mathcal{A}$ of bounded t-struct with stability function

$$Z_{\mathcal{A}} : K(\mathcal{A}) \to \mathbb{C}, \text{ s.th } (\mathcal{A}, Z_{\mathcal{A}}) \text{ have H-N property:}$$

Every object in $\mathcal{A}$ has an HN filtration by $Z_{\mathcal{A}}$-stable objects.

Proof: Construct (Z, P) from $(\mathcal{A}, Z_{\mathcal{A}})$.

$K(\mathcal{D}) = K(\mathcal{A})$, so Z and $Z_{\mathcal{A}}$ determine each other

For $\phi \in (0,1]$ let $\mathcal{P}(\phi) = \left\{ \begin{array}{l} Z_{\mathcal{A}} \text{ semistable} \\ \text{obj. of phase } \phi(E) = \phi \end{array} \right\}$

Extend this to $\phi \in \mathbb{R}$ by $\mathcal{P}(\phi + n) = \mathcal{P}(\phi)[n] \subset \mathcal{A}[n]$
$\phi \in (0,1]$

Then (1) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$ ✓
$Z(E) = m(E) e^{i\pi\phi}$ ✓

To check (2) For $\phi_1 > \phi_2$ $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$
(3) HN filtration

(2) is easy: For heart of bounded t-structure $\mathcal{A} \subset \mathcal{D}$ we know $\mathrm{Hom}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$ $k_1 > k_2$
and for stab. function on $\mathcal{A}$ we know $\mathrm{Hom}(E, F) = 0$ if
E s.s. of phase ϕ_1, F s.s of phase ϕ_2, $\phi_1 > \phi_2$.

(3) If $E \in \mathcal{D}$, $A_i \in \mathcal{A}[k_i]$ filtration by coh. objects.
$0 \hookrightarrow A_{i1} \hookrightarrow \dots \hookrightarrow A_{i,m_i} = A_i$ HN filtration in $\mathcal{A}$.
These can be put together to HN filtration of E:

Start with
$0 \to F_1 = A_{11}[k_1] \to F_2 = A_{12} \to \dots F_{m_1} = A_1[k_1]$

F_{m_1+i} is the cone of the composition
$A_{2i}[k_2] \to A_2[k_2] \to E_1[1]$ (extension of $A_2[k_2]$ by E_1)

Continue in this way.

$$\begin{tikzcd} E^0 \arrow[r] & E^1 \arrow[r] & E^2 \arrow[r] \arrow[d] & \dots \\ & & A_2[k_2] \arrow[ul] & \end{tikzcd}$$

<u>Example</u>: C smooth proj. curve, $\mathcal{D} = D^b(C)$

$$A = \mathrm{Coh}(C), \quad Z(E) = -\deg(E) + i\, r(E)$$

Z is a stab. function with H-N property.

The induced stability function has as semistable objects the shifts of slope semistable V.B and shapes of 0-dim torsion sheaves.