Higher Inductive Types

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V. Higher Inductive Types (HIT)

V.1. Motivation

(1) The Axiom of Choice holds trivially for type families $B : A \to \mathcal{U}$:

If $\underbrace{\text{for each } a : A \text{ we have } b : B\,a}$ then we have a $\underbrace{\text{dependent function mapping each } a : A \text{ into } B\,a}$.

$$\prod_{a:A} B\,a \xrightarrow{\quad id \quad} \prod_{a:A} B\,a$$

$$\boxed{\text{Functions in type theory are always given by a function rule!}}$$

(2) In Set Theory, the Axiom of Choice is applied on maps from a set into a set of non-empty sets: We only know the existence of an element in these target sets but cannot generate them with a function rule. Thus we obtain a function not described by a rule.

(3) The mere existence of an object in a type can be encoded in a Higher Inductive Type (HIT) $\equiv$ inductive type with constructors of objects & equalities between objects (& equalities of equalities ...)

V.2. Truncation of types to propositions :

$\Updownarrow$ smash operator in homotopy theory

$A : \mathcal{U} \rightsquigarrow \|A\| : \mathcal{U}$ with constructors $a : A \mapsto |a| : \|A\|$

$a, b : A \mapsto |a| =_{\|A\|} |b|$

- $\|A\|$ forgets "inner structure" of A and turns it into a proposition.
- Induction principle on $\|A\|$: $f : A \to B$ & $\prod_{x,y:A} f(x) =_B f(y) \Rightarrow \|A\| \to B$
- Not covered by Calculus of Inductive Constructions : inductive principle of HIT as axiom?

  $\Updownarrow$

  Computability?

V.3. Axiom of Choice : X set , $Y : X \to \mathcal{U}$ type family of sets $Y(x)$

$$(AC) \qquad \boxed{\prod_{x:X} \|Y(x)\| \longrightarrow \|\prod_{x:X} Y(x)\|}$$

"If we merely know that each $Y(x)$ is non-empty, then we merely know that there is a function $\prod_{x:X} Y(x)$."

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Lemma: There exists a type $X : \mathcal{U}$ and a family $Y : X \to \mathcal{U}$ such that each $Y(x)$ is a set, but (AC) is false.

Proof: Take $X :\equiv \sum_{A:\mathcal{U}} \| \mathbf{2} =_{\mathcal{U}} A \|$, $x_0 :\equiv \langle \mathbf{2}, |\mathrm{idpath}\,\mathbf{2}| \rangle : X$, $Y(x) :\equiv (x_0 =_X x)$ for each $x : X$.

(1) X is not a set:
For all $\langle A, p \rangle, \langle B, q \rangle : X$ we have $\langle A, p \rangle =_X \langle B, q \rangle \simeq (A \simeq B)$ by univalence and since p, q are objects in propositions.

$\Rightarrow (x_0 =_X x_0) \simeq (\mathbf{2} \simeq \mathbf{2}) \Rightarrow x_0 =_X x_0$ is not a proposition $\Rightarrow X$ is not a set.

(2) $Y(x) :\equiv (x_0 =_X x)$ is a set:
Do induction on $x : X \rightsquigarrow x :\equiv \langle A, p \rangle$. Then A is a set:
This claim is a proposition, hence we can do truncation induction on $p : \| \mathbf{2} =_{\mathcal{U}} A \|$ and use that $\mathbf{2}$ is a set.
$\Rightarrow \mathbf{2} \simeq A$ is a set (the set of bijections), and equivalent to $x_0 :\equiv \langle \mathbf{2}, |\mathrm{idpath}\,\mathbf{2}| \rangle =_X \langle A, p \rangle$.

(3) There is a function $\prod_{x:X} \| Y(x) \|$:
For every $\langle A, p \rangle : X$ we have $p : \| \mathbf{2} =_{\mathcal{U}} A \|$ and hence $\| Y(x) \| \equiv \| x_0 = \langle A, p \rangle \|$, by truncation induction on p.

On the other hand, $\| \prod_{x:X} Y(x) \| \to \mathrm{False}$: False is a proposition, so we can do truncation induction on $\| \prod_{x:X} Y(x) \|$ and assume $\prod_{x:X} Y(x) = \prod_{x:X} (x_0 =_X x)$.

But then, X is a proposition, hence a set – contradiction to (1).

Theorem (Diaconescu): $AC \Rightarrow LEM_{-1}$.

IV. 4. Set-quotients

A set, $R : A \to A \to \mathrm{Prop}$ family of propositions $=$ set-relation

Set-quotient A/R has constructors:

- $q : A \to A/R$, $a \mapsto [a] :\equiv q(a)$
- For each $a, b : A$ such that $R(a,b)$, an equality $[a] = [b]$.
- The 0-truncation functor: For all $x, y : A/R$, $r, s : x = y$ we have $r = s$.

• The 0-truncation functor: For all $x, y : A/R$, $r, s : x = y$ we have $r = s$.
($\Leftrightarrow$ the set-quotient A/R is a set)

Proposition: A/R is the set-coequalizer of the two projections $\sum_{a,b:A} R(a,b) \rightrightarrows A$, that is,

for any set B, $(A/R \to B) \simeq \sum_{f:A\to B} \prod_{a,b:A} R(a,b) \to (f(a) = f(b))$.

Proof: Given $f : A \to B$ such that $e : \prod_{a,b:A} R(a,b) \to (f(a) = f(b))$,

define $\bar{f} : A/R \to B$ by quotient induction: $\bar{f}([a]) :\equiv f(a)$

$r : R(a,b) \mapsto \bar{f}(e(r)) : \bar{f}([a]) = \bar{f}([b])$

$\langle f, e \rangle \mapsto \bar{f}$ is right-inverse to $\bar{f} \mapsto \langle \bar{f} \circ q, e \rangle$: $\langle f, e \rangle \mapsto \bar{f} \mapsto \langle \bar{f} \circ q, e \rangle = \langle f, e \rangle$.

For left-inverse, we need $\overline{g \circ q} = g$ for all $g : A/R \to B$. This follows from:

Claim: For all $x : A/R$, $\| \sum_{a:A} [a] = x \|$ (i.e., q is surjective).

Proof: Do quotient induction:
- $x \equiv [a]$ for some $a : A$: trivial
- $\| \sum_{a:A} [a] = x \|$ is a proposition $\Rightarrow$ path constructors automatically satisfied.

Now show $\overline{g \circ q} = g$ by quotient induction: • $[a] \equiv x$ for some $a : A$

$\Rightarrow g(x) = g([a]) = (g \circ q)(a) = \overline{g \circ q}([a]) = \overline{g \circ q}(x)$.

Calculus of Inductive Constructions + Martin-Löf equality + Univalence + Higher Inductive Types = HoTT / UF / HIT

univalent foundations