A brief introduction to Triangulated Categories

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1. Triangulated categories

In this section we introduce the axioms of a triangulated category and we derive some elementary properties from them.

Definition 1.1. — Let $\mathcal{C}$ be an additive category and let $T\colon \mathcal{C} \to \mathcal{C}$ be an additive auto-equivalence. A *triangle* in $\mathcal{C}$ with respect to T is a diagram of the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

A *morphism of triangles* is a commutative diagram of the form:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X' & \xrightarrow{u} & Y' & \xrightarrow{v} & Z' & \xrightarrow{w} & TX' \end{array}$$

Definition 1.2. — A *triangulated* category is a triple $(\mathcal{T}, T, \mathcal{D})$ where $\mathcal{T}$ is an additive category, $T\colon \mathcal{T} \to \mathcal{T}$ is an additive auto-equivalence and $\mathcal{D}$ is a class of candidate triangles, called *distinguished triangles*, satisfying the following axioms:

(TR_0) The class of distinguished triangles is closed under isomorphisms. Moreover, the candidate triangle:

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow TX$$

is distinguished.

(TR_1) For any morphism $f\colon X \to Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

(TR_2) Consider the two candidate triangles:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \tag{1}$$

and

$$Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY \tag{2}$$

Then, (1) is a distinguished triangle if and only if (2) is so.

(TR_3) For any commutative solid diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow Tf \\ X' & \xrightarrow{u} & Y' & \xrightarrow{v} & Z' & \xrightarrow{w} & TX' \end{array}$$

There exists a dotted arrow making the diagram commutative.

(TR_4) Assume we are given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ fitting into distinguished triangles:

$$X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow TX$$

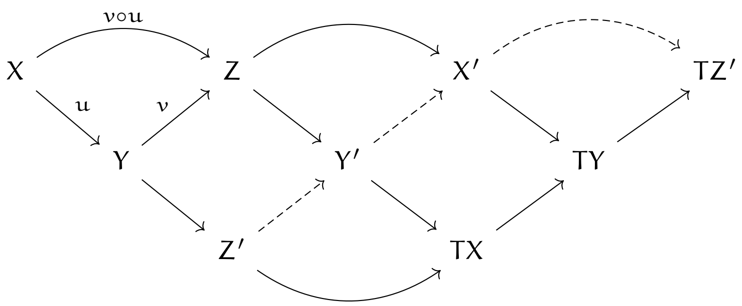
$$Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow TY$$

$$X \xrightarrow{v \circ u} Z \longrightarrow Y' \longrightarrow TX$$

then, there exists a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow TZ'$$

making the following diagram commutative:



We will often say that $(\mathcal{T}, T)$ or even just $\mathcal{T}$ is a triangulated category, omitting the auto-equivalence T and the class of distinguished triangles from the notation

Remark 1.3. — If $\mathcal{T}$ is a triangulated category and

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

is a distinguished triangle in $\mathcal{T}$, it follows from the axioms of a triangulated category that the compositions $v \circ u$, $w \circ v$ and $Tu \circ w$ are equal to the 0 morphism. Indeed, if we can consider the solid diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & TX \\
{\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle u} & & \vdots & & \downarrow{\scriptstyle \mathrm{id}_{TX}} \\
X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX
\end{array}$$

by the axiom (TR_3) there exists a dotted arrow making the diagram commutative. In particular, $v \circ u = 0$. Similarly, using axiom (TR_2) and (TR_3) one can show that the other compositions are zero.

Remark 1.4. — Given a triangulated category $(\mathcal{T}, T)$ one can easily see that the opposite category $\mathcal{T}^{op}$ inherits the structure of a triangulated category, with auto-equivalence given by the opposite of the quasi-inverse $(T^{-1})^{op} \colon \mathcal{T}^{op} \to \mathcal{T}^{op}$ and distinguished triangles of the form

$$Z \xleftarrow{\ u\ } Y \xleftarrow{\ v\ } X \xleftarrow{\ w\ } T^{-1}Z$$

such that the triangle

$$X \xrightarrow{\ v\ } Y \xrightarrow{\ u\ } Z \xrightarrow{-Tw} TX$$

is distinguished in $\mathcal{T}$.

Definition 1.5. — Let $\mathcal{T}$ be a triangulated category, let $\mathcal{A}$ be an abelian category and $H \colon \mathcal{T} \to \mathcal{A}$ be an additive functor. We say that H is *homological* (for $\mathcal{T}$) if, for every distinguished triangle

$$X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } TX$$

The sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

is exact in $\mathcal{A}$. Dually, a *cohomological functor* (for $\mathcal{T}$) is a functor $H \colon \mathcal{T}^{op} \to \mathcal{A}$ such that H is homological for $\mathcal{T}^{op}$.

Remark 1.6. — Let $\mathcal{T}$ be a triangulated category and $H \colon \mathcal{T} \to \mathcal{A}$ be a homological functor. Thanks to the axiom (TR_2) we see that the infinite sequence:

$$\cdots \longrightarrow H\left(T^{-1}(Z)\right) \longrightarrow H(X) \longrightarrow H(Y) \longrightarrow H(Z) \longrightarrow H(TX) \longrightarrow \cdots$$

is exact everywhere.

Proposition 1.7. — *Let $\mathcal{T}$ be a triangulated category, then for every object $A \in \mathcal{T}$, the functor*

$$\hom(A, -) \colon \mathcal{T} \to \mathsf{Ab}$$

is homological.

Proof. Given a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

we need to show that the sequence

$$\hom(A, X) \xrightarrow{u_*} \hom(A, Y) \xrightarrow{v_*} \hom(A, Z)$$

is exact. Clearly the composition $v_* \circ u_*$ is equal to zero. So let $f\colon A \to Y$ be a morphism such that $v \circ f\colon A \to Z$ is equal to zero. Then, we have a solid commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & TA & \xrightarrow{-\mathrm{id}_{TA}} & TA \\ {\scriptstyle f}\downarrow & & \downarrow & & \vdots & & \downarrow{\scriptstyle Tf} \\ Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX & \xrightarrow[-Tu]{} & TY \end{array}$$

The bottom row is a distinguished triangle by (TR_2), the top row by (TR_0) and (TR_2). Hence, by (TR_3) we can find a dotted map making the diagram commutative. Moreover, since T is fully-faithful, such a map is given by $Th\colon TA \to TX$ for exactly one map $h\colon A \to X$. Since the right square commutes, we have that $T(u \circ h) = T(f)$, which implies that $f = u \circ h$. Thus, $h \in \hom(A, X)$ is an element mapping to $f \in \hom(A, Y)$ and we are done. □

Corollary 1.8 (Two-out-of-three-property). — *Let us consider a morphism of distinguished triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' \end{array}$$

Then, if any two of the vertical morphisms f, g *and* h *are isomorphisms, so is the third.*

Proof. Without loss of generality we can assume that f and g are isomorphisms. For every $A \in \mathcal{T}$ we have a morphism of exact sequences:

$$\begin{array}{ccccccccc} \hom(A, X) & \longrightarrow & \hom(A, Y) & \longrightarrow & \hom(A, Z) & \longrightarrow & \hom(A, TX) & \longrightarrow & \hom(A, TY) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle Tg} \\ \hom(A, X') & \longrightarrow & \hom(A, Y') & \longrightarrow & \hom(A, Z') & \longrightarrow & \hom(A, TX') & \longrightarrow & \hom(A, TY') \end{array}$$

Since the rows are exact, by the Five Lemma we can conclude that h is an isomorphism. □

Corollary 1.9. — *Let* $\mathcal{T}$ *be a triangulated category and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

be a distinguished triangle in $\mathcal{T}$*. Then,* $u\colon X \to Y$ *is an isomorphism if and only if* Z *is isomorphic to the zero object.*

Proof. Let us consider the diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow & & \downarrow{\scriptstyle Tu} \\
Y & \xrightarrow[\mathrm{id}_Y]{} & Y & \longrightarrow & 0 & \longrightarrow & TY
\end{array}$$

Then, both rows of the diagram are distinguished triangles by assumption and axiom (TR_0). Then, by Corollary 1.8, we conclude that u is an isomorphism if and only if $Z \to 0$ is an isomorphism. $\square$

Exercise 1.10. — Let $\mathcal{T}$ be a triangulated category. Show that any triangle of the form:

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{0} TX$$

is isomorphic to a triangle of the form:

$$X \longrightarrow X \oplus Z \longrightarrow Z \xrightarrow{0} TX$$

2. Triangulated functors and Verdier quotient

Here, we introduce morphisms of triangulated categories, triangulated subcategories, and quotients.

Definition 2.1. — Let $(\mathcal{T}, T)$ and $(\mathcal{T}', T')$ be triangulated categories. A *triangulated functor* (or *exact functor*) from $\mathcal{T}$ to $\mathcal{T}'$ is an additive functor:

$$F\colon \mathcal{T} \to \mathcal{T}'$$

together with a natural isomorphism

$$\varphi\colon F \circ T \simeq T' \circ F$$

such that, for every distinguished triangle:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$$

in $\mathcal{T}$, the triangle:

$$\mathsf{F}X \xrightarrow{u} \mathsf{F}Y \xrightarrow{v} \mathsf{F}Z \xrightarrow{w} \mathsf{T}\mathsf{F}X$$

is distinguished in $\mathcal{T}'$.

Definition 2.2. — Let $\mathcal{T}$ be a triangulated category. A *triangulated subcategory* of $\mathcal{T}$ is a subcategory $\iota\colon \mathcal{C} \subset \mathcal{T}$ of $\mathcal{T}$ with the structure of a triangulated category, such that the inclusion functor ι is a triangulated functor.

Remark 2.3. — Let $\mathcal{T}$ be a triangulated category and let $\mathcal{C}$ be a full subcategory of $\mathcal{T}$. Then, $\mathcal{C}$ is a triangulated subcategory of $\mathcal{T}$ if and only if $\mathcal{C}$ is invariant under the functor T and for every distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathsf{T}X$$

in $\mathcal{T}$, with X and Y in $\mathcal{C}$, the object Z is isomorphic to an object of $\mathcal{C}$.

Lemma 2.4. — *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories and consider an adjoint pair of functors*

$$\mathcal{T} \underset{\mathsf{G}}{\overset{\mathsf{F}}{\underset{\perp}{\rightleftarrows}}} \mathcal{T}'$$

Then, F *is a triangulated functor if and only if* G *is so.*

Proof. See [Huy06, Proposition 1.41] □

Remark 2.5. — We can form a (large) 2-category of triangulated categories, denoted by Triang, with triangulated functors as morphisms and natural transformations as 2-morphisms. In particular, a triangulated functor $\mathsf{F}\colon \mathcal{T} \to \mathcal{T}'$ is said to be a *triangulated equivalence* if there exist a triangulated functor $\mathsf{G}\colon \mathcal{T}' \to \mathcal{T}$ such that

$$\mathsf{F} \circ \mathsf{G} \simeq \mathrm{id}_{\mathcal{T}'}, \quad \mathsf{G} \circ \mathsf{F} \simeq \mathrm{id}_{\mathcal{T}}.$$

By Lemma 2.4 we can conclude that $\mathsf{F}\colon \mathcal{T} \to \mathcal{T}'$ is a triangulated equivalence if and only if F is a triangulated functor and an equivalence of categories.

Example 2.6. — Let $\mathsf{F}\colon \mathcal{T} \to \mathcal{T}'$ be a triangulated functor. We define the *kernel* of F as the full subcategory $\ker(\mathsf{F}) \subset \mathcal{T}$ of $\mathcal{T}$ spanned by the objects $X \in \mathcal{T}$ such that $\mathsf{F}(X)$ is isomorphic to 0. Then, one can show that $\ker(\mathsf{F})$ is a triangulated subcategory of $\mathcal{T}$.

Example 2.7. — Similarly, if $\mathsf{H}\colon \mathcal{T} \to \mathcal{A}$ is a homological functor, we denote by $\ker(\mathsf{H})$, and call it the *stable kernel* of H the full subcategory of $\mathcal{T}$ consisting of those objects $X \in \mathcal{T}$ such that $\mathsf{H}(\mathsf{T}^i(X))$ is isomorphic to 0, for every i. One can show that $\ker(\mathsf{H})$ is a triangulated subcategory of $\mathcal{T}$.

2.8. — Let $\mathcal{T}$ be a triangulated category and let $\mathcal{C}$ be a full triangulated subcategory of $\mathcal{T}$. The *Verdier quotient* of $\mathcal{T}$ by $\mathcal{C}$ is a triangulated category $\mathcal{T}/\mathcal{C}$ together with a triangulated functor:

$$Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{C}$$

satisfying the following axioms

(1) The triangulated subcategory $\mathcal{C}$ is the kernel of F.

(2) For every triangulated functor $F\colon \mathcal{T} \to \mathcal{T}'$ such that $\mathcal{C}$ is contained in the kernel of F, there exists a unique triangulated functor $\tilde{F}\colon \mathcal{T}/\mathcal{C} \to \mathcal{T}'$ making the following diagram commutative

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{Q} & \mathcal{T}/\mathcal{C} \\ & \searrow^{F} & \downarrow{\scriptstyle \tilde{F}} \\ & & \mathcal{T}' \end{array}$$

Definition 2.9. — Let $\mathcal{T}$ be a triangulated category and let $H\colon \mathcal{T} \to \mathcal{A}$ be a homological functor. The *system $\mathcal{S}$ arising from the homological functor* H is the class of maps s such that $H(T^i(s))$ is an isomorphism for every integer i.

Theorem 2.10. — *Let $(\mathcal{T}, T)$ be a triangulated category and let $H\colon \mathcal{T} \to \mathcal{A}$ be a homological functor of $\mathcal{T}$. Then,*

(1) The system $\mathcal{S}$ arising from H is a multiplicative system.

(2) The Verdier quotient of $\mathcal{T}$ by $\ker(H)$ exists and is given by the (categorical) localization of $\mathcal{T}$ with respect to $\mathcal{S}$.

Sketch of the proof. Since the localization exists and has a universal property, it is enough to prove that the categorical localization $\mathcal{T}_{\mathcal{S}}$ is a triangulated category, that $Q_{\mathcal{S}}$ is a triangulated functor and that the functor arising from the universal property of the localization is triangulated.

Then, the universal property of the Verdier quotient will be satisfied since, given a map $u\colon X \to Y$ in $\mathcal{T}$, it is easy to see that $H(u[i])$ is an isomorphism for every i if and only if $H(\text{cone}(u)) = 0$, being H homological.

To check that the localization is triangulated, first we see that T defines an automorphism $T\colon \mathcal{T}_{\mathcal{S}} \to \mathcal{T}_{\mathcal{S}}$ by the rule

$$T(fs^{-1}) = T(f)T(s)^{-1}$$

We define distinguished triangles in $\mathcal{T}_{\mathcal{S}}$ as follows. Following the notation of A.3, let us consider a triangle of the form

$$X \overset{\alpha}{\dashrightarrow} Y \overset{\beta}{\dashrightarrow} Z \overset{\gamma}{\dashrightarrow} T(A)$$

in $\mathcal{T}_{\mathcal{S}}$. Then, using Ore condition repeatedly, we can represent the classes by morphisms fitting in the following diagram

$$\begin{array}{ccccccc} A' & \xrightarrow{u} & B' & \xrightarrow{v} & C' & \xrightarrow{w} & TA \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle id} \\ A & \overset{\alpha}{\dashrightarrow} & B & \overset{\beta}{\dashrightarrow} & C & \overset{\gamma}{\dashrightarrow} & TA \end{array}$$

where all the vertical morphisms are in $\mathcal{S}$. Therefore, we say that the (α, β, γ) is distinguished if and only if (u, v, w) defines a distinguished triangle in $\mathcal{T}$. As customary, we leave to the reader as an exercise that the axioms (TR_0) to (TR_4) are satisfied. □

3. The derived category of an abelian category

Let $\mathcal{A}$ be an abelian category, we denote by $\mathsf{Ch}(\mathcal{A})$ the *category of (co)-chain complexes* in $\mathcal{A}$ and by $\mathsf{K}(\mathcal{A})$ the *homotopy category* of $\mathsf{Ch}(\mathcal{A})$. Moreover, we denote by $\mathsf{K}^\star(\mathcal{A})$ the bounded above, bounded below or bounded subcategory of $\mathsf{K}(\mathcal{A})$, for $\star$ equal to $+$, $-$ or b. We denote by $\mathsf{D}(\mathcal{A})$ the localization of $\mathsf{K}(\mathcal{A})$ at the class of quasi isomorphisms and by $\mathsf{D}^\star(\mathcal{A})$ the corresponding bounded full subcategories.

3.1. — Recall that if $u\colon A \to B$ is a map of chain complexes in $\mathcal{A}$, the *mapping cone of* u, denoted by $\mathrm{cone}(u)$ is the chain complex given by

$$\mathrm{cone}(u)^i = A^{i+1} \oplus B^i$$

with differentials given by

$$d^i_{\mathrm{cone}(u)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ u^{i+1} & d_B^i \end{pmatrix}$$

Moreover, the cone comes equipped with natural maps of chain complexes $\tau\colon B \to \mathrm{cone}(u)$ and $\pi\colon \mathrm{cone}(u) \to A[1]$, where [1] denotes the *shift functor*.

3.2. — Since the shift functor preserves homotopies, it descend to an endofunctor $[1]\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ which is an equivalence as well. We define the class of *distinguished triangles* in $\mathsf{K}(\mathcal{A})$ as the class of diagrams isomorphic (in $\mathsf{K}(\mathcal{A})$) to a diagram of the form

$$A \xrightarrow{u} B \xrightarrow{\tau} \mathrm{cone}(u) \xrightarrow{\pi} A[1] \tag{3}$$

Proposition 3.3. — *Let* $u\colon A \to B$ *be a morphism of complexes and let us consider the diagram*

$$A \xrightarrow{u} B \xrightarrow{\tau} \mathrm{cone}(u) \xrightarrow{\pi} A[1]$$

Then, there exists a morphism $\nu\colon A[1] \to \mathrm{cone}(\tau)$ *which is an isomorphism in* $\mathsf{K}(\mathcal{A})$ *and such that the following diagram commutes in* $\mathsf{K}(\mathcal{A})$.

$$\begin{array}{ccccccc}
B & \xrightarrow{\tau} & \mathrm{cone}(u) & \xrightarrow{\pi} & A[1] & \xrightarrow{u[1]} & B[1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
B & \xrightarrow[u]{} & \mathrm{cone}(u) & \xrightarrow[\tau]{} & \mathrm{cone}(\tau) & \xrightarrow[\pi]{} & B[1]
\end{array}$$

Proof. See [Huy06, Proposition 2.16]. □

Theorem 3.4. — *Let* $\mathcal{A}$ *be an abelian category and let* $\mathsf{K}(\mathcal{A})$ *be the associated homotopy category. Then, the shift functor and the class of distinguished triangles defined above give* $\mathsf{K}(\mathcal{A})$ *the structure of a triangulated category.*

Sketch of the proof. To show (TR_0) notice that $\mathrm{cone}(\mathrm{id}_A)$ is a split exact complex and in particular is isomorphic to the zero object in $\mathsf{K}(\mathcal{A})$. To show axiom (TR_1) is enough, for a given map $u\colon A \to B$ in $\mathsf{K}(\mathcal{A})$, to take a representative in $\mathsf{Ch}(\mathcal{A})$ and consider the induced diagram (3). Axiom (TR_2) follows from 3.3. To show axiom (TR_3), we can consider a diagram of the form

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{\tau} & \mathrm{cone}(u) & \xrightarrow{\pi} & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \xrightarrow{u'} & B' & \xrightarrow{\tau'} & \mathrm{cone}(u') & \xrightarrow{\pi'} & A'[1]
\end{array}$$

And the dotted arrow follows from functoriality of the cone in $\mathsf{Ch}(\mathcal{A})$. We omit the proof of (TR_4) and refer to [Wei95, Proposition 10.2.4]. □

Corollary 3.5. — *Let* $\mathcal{A}$ *be an abelian category. Then, the full subcategories* $\mathsf{K}^\star(\mathcal{A})$ *of* $\mathsf{K}(\mathcal{A})$ *are triangulated.*

Proof. By Remark 2.3 it suffices to show that the boundedness conditions are preserved under shift functor and under taking cones, which is immediate from the definitions. □

Remark 3.6. — Let $\mathcal{A}$ be an abelian category and let us consider a distinguished triangle of the form

$$A \xrightarrow{u} B \xrightarrow{\tau} \mathrm{cone}(u) \xrightarrow{\pi} A[1]$$

Then, one can show that τ and π fit in a short exact sequence

$$0 \longrightarrow B \xrightarrow{\tau} \mathrm{cone}(u) \xrightarrow{\pi} A[1] \longrightarrow 0$$

which, classically, induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^i(B) \longrightarrow H^i(\mathrm{cone}(u)) \longrightarrow H^i(A[1]) \xrightarrow{\partial} H^{i+1}(B) \longrightarrow \cdots$$

One can show that indeed $\partial = H^i(u[1])$ under the natural isomorphism $H^{i+1}(B) \simeq H^i(B[1])$ and so that the long exact sequence is induced by the functor H^0. In particular, the functor H^0 is homological in the sense of Definition 1.5.

Proposition 3.7. — *Let $\mathcal{A}$ be an abelian category and let $H^0\colon K(\mathcal{A}) \to \mathcal{A}$ be the 0-th cohomology functor. Then, H^0 is an homological functor. In particular, $\ker(H^0)$ is a triangulated full subcategory of $K(\mathcal{A})$.*

Proof. This follows from the discussion in Remark 3.6 and by Example 2.7. □

Theorem 3.8. — *Let $\mathcal{A}$ be an abelian category. Then, the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is the Verdier quotient of $K(\mathcal{A})$ with respect to $\ker(H^0)$. In particular, $D(\mathcal{A})$ is a triangulated category.*

Proof. The class of quasi isomorphisms in $K(\mathcal{A})$ forms a system arising from the homological functor H^0. Therefore, we can conclude by Theorem 2.10. □

Corollary 3.9. — *Let $\mathcal{A}$ be an abelian category. Then the bounded derived categories $D^\star(\mathcal{A})$ are triangulated full subcategories of $D(\mathcal{A})$.*

Remark 3.10. — Given an abelian category $\mathcal{A}$ and a short exact sequence of chain complexes in $\mathcal{A}$:

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

even though we can associate a long exact sequence in cohomology to it, there might be no map from C to $A[1]$ in $K(\mathcal{A})$.

However, one can consider the following distinguished triangles:

$$A \xrightarrow{u} B \xrightarrow{\tau} \mathrm{cone}(u) \xrightarrow{\pi} A[1]$$

and

$$\mathrm{cone}(u) \xrightarrow{\pi} A[1] \xrightarrow{\tau'} \mathrm{cone}(\pi) \xrightarrow{\pi'} \mathrm{cone}(u)[1]$$

Then, applying H^0 to the last triangle we get a long exact sequence:

$$\cdots \longrightarrow H^i(A[1]) \longrightarrow H^i(\mathrm{cone}(\pi)) \longrightarrow H^i(\mathrm{cone}(u)[1]) \longrightarrow \cdots$$

Then, one can see that $\mathrm{cone}(\pi)$ is quasi isomorphic to $B[1]$ and that $\mathrm{cone}(u)$ is quasi isomorphic to C, so we recover the long exact sequence associated to the original short exact sequence. Moreover, notice that the quasi isomorphism $\varphi\colon \mathrm{cone}(u) \to C$ is invertible in $D(\mathcal{A})$, so that we get an exact triangle in $D(\mathcal{A})$, given by

$$A \xrightarrow{u} B \xrightarrow{\tau} C \xrightarrow{\pi\varphi^{-1}} A[1]$$

A. Localization of categories and calculus of fraction

The localization of a category $\mathcal{C}$ at a class of morphism $\mathcal{S}$ is a procedure that allows us to "formally invert" all the morphisms in $\mathcal{S}$ in a suitable sense. The localization always exists (in a big enough universe) but in this small recollection we are mainly concerned with localizations with respect to *multiplicative systems of morphisms*, that allow a slightly better control on the morphisms in the localization.

Definition A.1. — Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. The *localization of* $\mathcal{C}$ *with respect to* $\mathcal{S}$ is a couple $(\mathcal{C}_{\mathcal{S}}, Q_{\mathcal{S}})$ where $\mathcal{C}_{\mathcal{S}}$ is a category and

$$Q_{\mathcal{S}} \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$$

is a functor, satisfying the following properties.

(1) For every map $s \in \mathcal{S}$, the image $Q_{\mathcal{S}}(s)$ is an isomorphism in $\mathcal{C}_{\mathcal{S}}$.

(2) For every functor $F \colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism in $\mathcal{D}$, there exists a functor $\tilde{F} \colon \mathcal{C}_{\mathcal{S}} \to \mathcal{D}$ such that $\tilde{F} \circ Q_{\mathcal{S}}$ is naturally isomorphic to F.

Definition A.2. — Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a collection of morphisms in $\mathcal{C}$. We say that $\mathcal{S}$ is a *multiplicative system* in $\mathcal{C}$ if the following axioms hold:

(1) The class $\mathcal{S}$ is closed under composition and contains all the identity morphisms.

(2) (Ore condition) If $t \colon Z \to Y$ is a morphism in $\mathcal{S}$, then for every morphism $g \colon X \to Y$ in $\mathcal{C}$, there exist dotted arrows making the following diagram commutative:

$$\begin{array}{ccc} W & \overset{f}{\dashrightarrow} & Z \\ {\scriptstyle s}\,\vdots\downarrow & & \downarrow{\scriptstyle t} \\ X & \underset{g}{\longrightarrow} & Y \end{array}$$

Dually, if $s \colon X \to Z$ is a map in $\mathcal{S}$, then for every map $f \colon X \to Y$ in $\mathcal{C}$, there exist dotted arrows making the following diagram commutative:

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle s}\downarrow & & \vdots\downarrow{\scriptstyle t} \\ Z & \underset{g}{\dashrightarrow} & V \end{array}$$

(3) (Cancellation) If $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are morphisms in $\mathcal{C}$, then the following two conditions are equivalent:

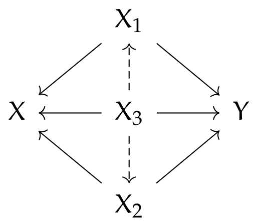
- There exist a map $s \colon Y \to Z$ in $\mathcal{S}$ such that $sf = sg$.
- There exists a map $t \colon W \to X$ in $\mathcal{S}$ such that $ft = gt$.

A.3. — Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a multiplicative system in $\mathcal{S}$. A *left fraction* from X to Y in $\mathcal{C}$ with respect to $\mathcal{S}$ is a diagram of the form

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

such that s is an element of $\mathcal{S}$.

Given left fractions $X \leftarrow X_1 \rightarrow Y$ and $X \leftarrow X_2 \rightarrow Y$ we say that they are equivalent if there exists a left fraction $X \leftarrow X_3 \rightarrow Y$ fitting in a commutative diagram:



If X and Y are objects in $\mathcal{C}$, we write $\hom_{\mathcal{S}}(X, Y)$ for the collection of equivalence classes of left fractions from X to Y. Notice that this is not a set a priori. An element in $\hom_{\mathcal{S}}(X, Y)$ will be denoted by a dotted arrow $X \overset{\gamma}{\dashrightarrow} Y$ and we will write $fs^{-1} \colon X \leftarrow X' \rightarrow Y$ when we specify an element in the equivalence class γ.

Theorem A.4 (Gabriel-Zisman). — *Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a multiplicative system of morphisms in $\mathcal{C}$. Then,*

(1) There exists a (possibly large) category $\mathcal{S}^{-1}\mathcal{C}$ with the same objects as $\mathcal{C}$ and with classes of maps given by

$$\hom_{\mathcal{S}^{-1}(\mathcal{C})}(X, Y) = \hom_{\mathcal{S}}(X, Y)$$

(2) The category $\mathcal{S}^{-1}\mathcal{C}$ is a localization of $\mathcal{C}$ with respect to $\mathcal{S}$.

Proof. See [Wei95, Theorem 10.3.7] □

Proposition A.5. — *Let $\mathcal{C}$ be an additive category and let $\mathcal{S}$ be a multiplicative system in $\mathcal{C}$. Then, the localization of $\mathcal{C}$ with respect to $\mathcal{S}$ is an additive category and the quotient functor*

$$Q_{\mathcal{S}} \colon \mathcal{C} \rightarrow \mathcal{C}_{\mathcal{S}}$$

is an additive functor.

Proof. See [Wei95, Corollary 10.3.11] □

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