These notes are the notes accompanying the talk given in the reading seminar on the Fourier Mukai Transform given on November 22nd 2017. Throughout all schemes and varieties will be considered to be over an algebraically closed field of characteristic zero unless otherwise stated. The main references are the books of Huybrechts [1] and Polishchuck [2].

History and motivation

Abelian varieties are a very well studied area of classical mathematics. They were first studied in the 19th century by Abel and Jacobi when they were studying elliptic integrals. They constructed an abelian surface which we now know to the the Jacobian of a hyperelliptic curve of genus 2. Through the 19th century they were studied further by Picard, Frobenius and Poincaré. Then in the 1920's Lefschetz constructed abelian varieties as complex tori and produced a lot of general results and finally in the 1940's Andre Weil formalised the study of abelian varieties into the modern language of algebraic geometry. Abelian varieties have a very rigid structure as we will hopefully see through this talk. In particular they provide an excellent testing ground for new results and a rich source of examples.

Basic definitions

Definition 1. *An abelian variety is a commutative group object in the category of complete varieties.*

This is the most general possible definition of an abelian variety as given in [2] and is a little misleading as the fact that an abelian variety is commutative is a consequence rather than a requirement. Let us now make this definition a little more precise.

Definition 2. *A variety is an irreducible and reduced scheme of finite type. A variety X is complete if for any other variety Y the projection $X \times Y \to Y$ is a closed map (i.e. it maps closed sets to closed sets).*

Definition 3. *A group scheme G is a group object in the category of schemes. G is equipped with morphisms $m : G \times G \to G, i : G \to G$ and $e : spec(k) \to G$ which satisfy the usual group axioms.*

Now using the above definitions we define an abelian variety to be;

Definition 4. *An abelian variety is a group scheme which is complete as a variety.*

The group law on an abelian variety A is usually written additively so we write $m(a, b) = a + b$ for $a, b \in A$, $i(a) = -a$ for the inverse of a and $e = 0 \in A$ for the identity. Then for any closed point $a \in A$ we have a map given by;

$$t_a : A \to A \text{ such that for all } b \in A, b \mapsto a + b.$$

We call this map translation by a. Moreover since t_a is an isomorphism for any closed point of A we find that any abelian variety is non-singular.

Definition 5. *Let A and B be abelian varieties then a map $f : A \to B$ is a homomorphism of abelian varieties is $f(e_A) = e_B$.*

Then some of the rigidity of the structure of abelian varieties can be seen by how maps on abelian varieties are constructed.

Lemma 6. *If A and B are abelian varieties then any map $f : A \to B$ is the composition of a translation and some homomorphism, say h, so we can write$f = t_{f(e_A)} \cdot h$.*

We can now use this important result to provide an easy proof to the following lemma,

Lemma 7. *The group structure of an abelian variety is commutative .*

Proof. First we note that if i is the inversion morphism for a given abelian variety and e is the identity element then $i(e) = e$. Hence $t_{i(e)} = t_e$ and $t_e \cdot h = h$ by the above lemma for some homomorphism h. Since i is also a homomorphism we have that the group structure on A is commutative.

□

A further example of a homomorphism of abelian varities is the map $[n]_A : A \to A$ such that $a \mapsto n\dot{a}$ for $n \in \mathbb{Z}$. We call this homomorphism multiplication by n.

So we have seen so far that any abelian variety is smooth and the underlying group is commutative. We now consider the special case when we set $k = \mathbb{C}$.

The link between abelian varieties and complex tori.

For this section we consider all abelian varities to be over the complex numbers. Then the associated complex manifold to any ableian variety is a compact, complex lie group. The following definition shows that over $\mathbb{C}$ we can realise any abelian variety as a complex tori.

Definition 8. *A complex torus is a connected compact, complex lie group.*

This can also be seen from another perspective.

Theorem 9 (GAGA)**.** *There is a functor from proper $\mathbb{C}-$schemes to compact Hausdorff $\mathbb{C}-$analytic spaces*

$$X \mapsto X^{an},$$

which is fully faithful.

A consequence of this is that the category of abelian varieties of dimension n over $\mathbb{C}$ sits fully faithfully inside the category of complex tori of dimension n. When the dimension is one, all abelian varieties are realisable as complex tori, however when the dimension is at least 2 there are many complex tori which are not abelian varieties.

Line bundles on abelian varieties.

Before we discuss line bundles in detail we first state an important result that displys some of the rigidity of abelian varieties.

Lemma 10 (Rigidity Lemma)**.** *Let X be a complete variety and let Y, Z be arbitary varieties. Then assume the map $f : X \times Y \to Y$ contracts $X \times y$ for some point $y \in Y$ to a point in Z. Then f is the composition of the projection $X \times Y \to Y$ and some morphism $h : Y \to Z$.*

For the proof of this Lemma see [2].

Proposition 11 (See-Saw principle.)**.** *Let X be a complete irreducible variety and let T be an integral scheme such that L is a line bundle on $X \times T$. Suppose there exists a line bundle $L_t := L_{|X \times t}$ which is trivial for all closed points $t \in T$, then there exists a line bundle M on T such that*

$$L \cong p^*M.$$

This is an important statement and in particular shows the following;

Remark 12.

1. If L is trivial on at least one fiber of the projection of $X \times T$ then L is trivial.

2. Suppose that L and L' are both line bundles on $X \times T$ such that $L_t = L'_t$ for all closed points $t \in T$ then

$$L \cong L' \otimes p^*M \text{ for some line bundle M on T .}$$

Example 13. *Let A be an abelian variety and L a line bundle on A. Then $m^*L = q^*L \otimes p^*L$ if and only if $t_a^*L \cong L$ for all $a \in A$. Here p and q are the projection maps given by $p : A \times A \to A$ and $q : A \times A \to A$.*

Line bundles that satisfy the property $t_a^*L \cong L$ are called translation invariant line bundles and are a very important class of line bundles on abelian varieties which we will see more of later. The next theorem we wish to discuss is a cornerstone result in the study of abelian varieties. It was first published by Lang in 1959 and is in part due to earlier work of Wiel. In particular this theorem gives us nice conditions on the product of 3 irreducible complete varieties for line bundles to be trivial.

Theorem 14 (Theorem of the square.)**.** *Let X, Y and Z be irreducible, complete varieties and let $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$ be closed points. Then a line bundle on $X \times Y \times Z$ is trivial if and only if the restrictions*

$$L_{|X\times Y\times z_0}, L_{|X\times y_0\times Z} \text{ and } L_{|x_0\times Y\times Z}$$

are trivial.

Proof. First we note that the restriction of L to the product $X \times Y \times Z$ is trivial over the fiber at x_0 so we just have to show that it is trivial for all other points $x \in X$. Connecting x_0 to any point $x \in X$ describes a complete curve so it is enough to reduced to the case when X is a curve. First we prove the result for when X is a smooth curve. For every $x \in X$ let $C \subset X$ be the smooth curve passing through x and x_0. Let $\tilde{C}$ be the normalisation of C and if we assume that the theorem holds on $\tilde{C} \times Y \times Z$ then the pull back of a line bundle L to $C \times Y \times Z$ is trivial. This shows that $L_{|x\times Y\times Z}$ is trivial. So for any points $(x, z) \in X \times Z$ the restriction $L_{|x\times Y\times z}$ is also trivial. Finally since Y is complete the line bundle L is simply the pull-back of a line bundle L' on $X \times Z$ and hence $L' \cong L_{|X\times y\times Z}$ is trivial. For the proof of the case when X is not smooth see [2]. ☐

The following remark shows a useful consequence of this theorem.

Remark 15. *Let A be an abelian variety and consider the product $A \times A \times A$ along with projections π_i : $A\times A\times A$ for $i = 1, 2, 3$. The group law on $A\times A\times A$ is given as $m : A\times A\times A \to A$ where $(a, b, c) \mapsto a+b+c$. Then there exists a map $m_{i,j} : A \times A \times A \to A$ given by the composition $m \cdot (\pi_i, \pi_j)$ such that we have the following isomorphism*

$$m^*L \otimes \pi_1^*L \otimes \pi_2^*L \otimes \pi_3^*L \cong m_{12}^*L \otimes m_{13}^*L \otimes m_{23}^*L.$$

Theorem 16 (Theorem of the square)**.** *Let A be an abelian variety then for all points $a, b \in A$ we have*

$$t_{a+b}^*L \otimes L \cong t_a^*L \otimes t_b^*L \text{ for all line bundles } L \text{ on } A.$$

The result in the above theorem is obtained by pulling back the isomorphism in Remark 15 along the map $A \to A \times A \times A$ where $c \mapsto (c, a, b)$.

Definition 17. *We now define the (very important) map*

$$\phi_L : A(k) \to \mathrm{Pic}(A), a \mapsto t_a^*L \otimes L^{-1}.$$

The above theorem says this map is a homomorphism.

We now want to introduce the Picard functor for abelian varieties. First we recall that for any variety X the Picard functor is a functor which for any other variety T associated the group of all equivalence classes of line bundles on $X \times T$, where two line bundles are equivalent if they are isomorphic up to tensoring by the pull back of a line bundle on T. If X is a projective variety then the Picard functor is represented by the algebraic group $\mathrm{Pic}(X)$.

Definition 18. $\mathrm{Pic}(X)$ *is the group of all isomorphism classes of line bundles / invertible sheaves on X.*

Since for any abelian variety A, $\mathrm{Pic}(A)$ is an algebraic group we can consider the subgroup $\mathrm{Pic}^0(A)$ which consist of all translation invariant line bundles on A.

$$\mathrm{Pic}^0(A) := \{L \in \mathrm{Pic}(A) : t_a^*L \cong L, \text{ for all } a \in A\}.$$

The subgroup $\mathrm{Pic}^0(A)$ represents all line bundles on A which are topologically equivalent to zero i.e. line bundles whose the first Chern class vanishes. Another nice property of translation invariant line bundles is that if they are non-trivial then they have trivial cohomology. This is shown by the following theorem.

Theorem 19. *Let A be an abelian variety then for any non-trivial $L \in \mathrm{Pic}^0(A)$ we have*

$$H^*(A, L) = 0.$$

Ampleness of line bundles

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In this section of the talk we will discuss some requirements of line bundles on ableian varieties to be ample and some properties of ample line bundles on abelian varieties. Let A be an abelian variety and L a line bundle on A. Recall from the previous section that there is a map $\phi_L : A(k) \to \mathrm{Pic}^0(A)$. Then we call the subgroup $K(L) \subset A(k)$ the kernel of the map ϕ_L and in particular $K(L) = A$ if and only if $L \in \mathrm{Pic}^0(A)$.

Proposition 20. *Let L be a line bundle on an abelian variety A such that $H^0(A, L) \neq 0$ then $L_{|K(L)}$ is trivial.*

Theorem 21. *A line bundle L on an abelian variety A is ample if and only if there exists some $n > 0$ such that $H^0(A, L^n)$ and $K(L)$ is finite.*

Theorem 22. *Every abelian variety is projective.*

Proof. To prove this claim we give a sketch of the proof from [2] where we see it is enough to find an effective divisor on A such that the complement of A by D is open affine. Choose a effective divisor D' and an open affine chart say U' such that $A = U' \cup D'$. Now select a non-zero function $f \in \mathcal{O}(U')$ that vanishes on the intersection of D' and U'. Let zf be the zero divisor of f and let $\bar{zf}$ be its closure in A. Then U is affine if the line bundle $L = \mathcal{O}_A(D)$ is ample. To show this we use the above theorem and show that the kernal $K(L)$ is finite. To show this we note that by translating if necessary U contains 0. Let $B \subset K(L)$ bf the connected component of zero then the restriction of $\mathcal{O}(D)$ to B is trivial. Since B is not contained in D the intersection of B and D is empty so B is a complete variety contained in an affine variety so B is a point. Hence we conclude that $K(L)$ is finite.

□

Before we proceed to the next section it is worth noting that all the above results for line bundles on abelian varieties are generalisable to complex tori. Over the complex numbers line bundles on complex tori are described by their Appel-Humbert datum and this allows us to describe the map ϕ_L and its kernel $K(L)$ in a similar way to above. For more details and for general results on line bundles on complex tori see [2] and [3].

The dual abelian variety

Throughout this section let A be an abelian variety and let L be a ample line bundle on A. Recall from the previous section if L is ample then the map ϕ_L is surjective and its kernel $K(L)$ is finite. We now want to introduce the dual of an abelian variety. Some of the material covered in this section requires more background knowledge than previous sections particularly the construction of quotients of group schemes by the action of a finite subgroup scheme. For full details see [2] or [3].
Now Let A be an abelian variety then we define the dual abelian variety $\hat{A}$ to be the quotient of A by the action of $K(L)$.

Theorem 23. *Let A be an abelian variety and L an ample line bundle on A then the quotient $A/K(L)$ exists and is an abelian variety.*

Sketch of proof. By definition A is an abelian group scheme so to prove the theorem we first need to prove that $K(L)$ is a finite subgroup scheme of A and that the quotient by the action is well defined. For full details of this step see [2]. Once we know this we show that the dimension of the quotient is equal the dimension of A and since the chartersitic of the ground field is zero all group schemes will be smooth hence $A/K(L)$ is a variety. The final step is to show that its is also complete. □

Definition 24. *For any abelian variety with an ample line bundle L the dual variety is given by*

$$\hat{A} = A/K(L).$$

Another approach to define the dual abelian variety (as discussed in [1]) is to consider the exponential sequence associated to an abelian variety A,

$$0 \to \mathbb{Z} \to \mathcal{O}_A \to \mathcal{O}_A^* \to 0.$$

This is an short exact sequence so we apply cohomology to it to obtain a exact sequence

$$H^1(A, \mathbb{Z}) \to H^1(A, \mathcal{O}) \to H^1(A, \mathcal{O}^*) \to H^2(A, \mathbb{Z}) \to H^2(A, \mathcal{O}).$$

We then define the dual abelian variety to be the quotient

$$\hat{A} = H^1(A, \mathcal{O})/H^1(A, \mathbb{Z}).$$

Claim 25. $\hat{A} = \mathrm{Pic}^0(A)$.

A sketch of a proof for the above claim is as follows.

Proof. First we note that $H^1(A, \mathcal{O}^*) = \mathrm{Pic}(A)$ and the map $c_1 : H^1(A, \mathcal{O}^*) \to H^2(A, \mathbb{Z})$ is the first Chern map. Then $\hat{A} \subset \mathrm{Pic}^0(A)$ since the induced action on $H^1(A, \mathcal{O}$ is trivial so every line bundle in $\hat{A}$ is translation invariant hence is contained in $\mathrm{Pic}^0 A$. Conversely for any $L \in \mathrm{Pic}^0(A)$ we have $i^*(L) = L^{-1}$ hence $c_1(L) = -L$ and $i_{|H^2(A,\mathbb{Z})} = id$. Finally since $H^2(A, \mathbb{Z})$ is torsion free $c_1(L) = 0$ hence $L \in \hat{A}$. □

The final property of abelian varieties I would like to mention is that on the product of an abelian variety with its dual there exists a uniquely determined line bundle. To make this more precise we have the following theorem.

Theorem 26 (Universal property of abelian varieties)**.** *Let A be an abelian variety then there is a uniquely determined line bundle $\mathcal{P}$ on $A \times \hat{A}$ called the Poincaré bundle such that;*

1. $\mathcal{P}_{|A\times\{y\}} \in \mathrm{Pic}^0(A \times \{y\})$ *for all* $y \in \hat{A}$
2. $\mathcal{P}_{|\{e\}\times\hat{A}}$ *is trivial.*

The line bundle $\mathcal{P}$ corresponds to the identity on $\hat{A}$.

Bibliography

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