The Derived Category of an Abelian Category

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October 18, 2017

Throughout, let $\mathcal{A}$ be an abelian category.

1 Basic Definitions

Definition 1.1. A *complex* $A^\bullet$ in $\mathcal{A}$ is a sequence of objects and morphisms

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{d^{i+1}} \xrightarrow{d^{i+1}} \cdots,$$

with $d^i \circ d^{i-1} = 0, \forall i \in \mathbb{Z}$.

Example 1.2. Let V be a vector space over some field. Then the sequence

$$\cdots \to \Omega^{k-1}V \xrightarrow{d} \Omega^k V \xrightarrow{d} \Omega^{k+1}V \to \cdots,$$

where d is the exterior derivative, is a complex. The d^i in an arbitrary complex are sometimes called differentials by analogy.

Example 1.3. All exact sequences are complexes.

Definition 1.4. Given complexes $A^\bullet$ and $B^\bullet$, a morphism $f : A^\bullet \to B^\bullet$ is a collection of morphisms $f^i : A^i \to B^i$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \xrightarrow{d_B^{i+1}} & \cdots
\end{array}$$

commutes.

The complexes and their morphisms form a category, $\mathrm{Kom}(\mathcal{A})$. This is an abelian category, with the zero object, kernels, etc. being as expected. We have an inclusion of categories $\mathcal{A} \subset \mathrm{Kom}(\mathcal{A})$ given by sending the object $A \in \mathcal{A}$ to the complex with $A^0 = A, A^i = 0, i \neq 0$.

Definition 1.5. Let $A^\bullet$ be a complex. The shifted complex $A^\bullet[1]$ is the complex given by $(A^\bullet[1])^i = A^{i+1}, d^i_{A[1]} = -d^{i+1}_A$.

We also get shifted morphisms $f[1] : A^\bullet[1] \to B^\bullet[1]$, given by $f[1]^i = f^{i+1}$.

This shifting is functorial, and indeed gives an equivalence of categories.

Definition 1.6. Given a complex $A^\bullet$, the *ith cohomology* is $H^i(A^\bullet) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$.

For a complex morhphism $f : A^\bullet \to B^\bullet$, there are induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$, given by $[a] \mapsto [f^i(a)]$. (That this is well-defined comes from the definitions.)

Definition 1.7. A complex morphism $f : A^\bullet \to B^\bullet$ is a *quasi-isomorphism*, or qis, if $\forall i \in \mathbb{Z}$, $H^i(f)$ is an isomorphism.

Example 1.8. Quasi-isomorphisms need not be invertible as complex morphisms, as illustrated below:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

(The morphism not given are the obvious ones). Calculating the cohomology shows that this is a quasi-isomorphism, but it is clearly not invertible.

Definition 1.9. Let $f, g : A^\bullet \to B^\bullet$ be complex morphisms. We say f and g are *homotopic*, $f \sim g$, if $\exists h^i : A^i \to B^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

This is an equivalence relation; we have $f \sim g \Rightarrow H^i(f) = H^i(g), \forall i$. Also, if we have morphisms $f : A^\bullet \to B^\bullet, g : B^\bullet \to A^\bullet$ with $f \circ g \sim \mathrm{id}_B, g \circ f \sim \mathrm{id}_A$, then f, g are quasi-isomorphisms, and $H^i(f)^{-1} = H^i(g)$.

Definition 1.10. The *homotopy category* $K(\mathcal{A})$ is the category with $\mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A})), \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$.

2 The Mapping Cone

Definition 2.1. Let $f : A^\bullet \to B^\bullet$ be a complex morphism. The *mapping cone* is the complex $C(f)$, with $C(f)^i = A^{i+1} \oplus B^i$,

$$d^i_{C(f)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

(A quick calculation shows that this is indeed a complex).

The mapping cone comes with two canonical morphisms: $\tau : B^\bullet \to C(f)$, given by the injection $B^i \to A^{i+1} \oplus B^i$, and $\pi : C(f) \to A^\bullet[1]$, given by the projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$.

Proposition 2.2. *With the mapping cone, we can complete commutative diagrams as follows:*

$$\begin{array}{ccccccc} A_1^\bullet & \xrightarrow{f_1} & B_1^\bullet & \longrightarrow & C(f_1) & \longrightarrow & A_1^\bullet[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ A_2^\bullet & \xrightarrow{f_2} & B_2^\bullet & \longrightarrow & C(f_2) & \longrightarrow & A_2^\bullet[1] \end{array}.$$

Proposition 2.3. *Let $f : A^\bullet \to B^\bullet$ be a complex morphism, and $C(f)$ the mapping cone. Then there exists a morphism $g : A^\bullet[1] \to C(\tau)$, isomorphic in $K(\mathcal{A})$, such that the diagram*

$$\begin{array}{ccccccc} B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A^\bullet[1] & \xrightarrow{-f} & B^\bullet[1] \\ = \downarrow & & = \downarrow & & g \, \vdots & & \downarrow = \\ B^\bullet & \xrightarrow{\tau} & C(f) & \xrightarrow{\tau_\tau} & C(\tau) & \xrightarrow{\pi_\tau} & B^\bullet[1] \end{array}$$

commutes up to homotopy.

The morphism g required in the proof is the morphism $A^{i+1} \to B^{i+1} \oplus A^{i+1} \oplus B^i$ given by $(-f^{i+1}, \mathrm{id}, 0)$.

Proposition 2.4. *Let $f : A^\bullet \to B^\bullet, g : C^\bullet \to B^\bullet$ be complex morphisms, with f a quasi-isomorphism. Then there exists a diagram*

$$\begin{array}{ccc} C_0^\bullet & \xrightarrow{qis} & C^\bullet \\ \downarrow & & \downarrow g \\ A^\bullet & \xrightarrow{f} & B^\bullet \end{array}$$

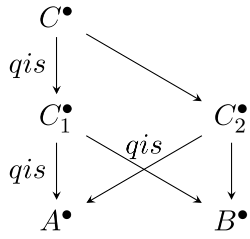
which commutes up to homotopy.

3 The Derived Category

The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is given in two parts. Firstly, the objects are the complexes in $\mathcal{A}$. The morphisms $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ are equivalence classes of diagrams

$$\begin{array}{ccc} C^\bullet & & \\ qis \downarrow & \searrow & \\ A^\bullet & & B^\bullet \end{array}$$

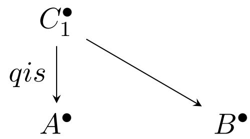
where two such diagrams are equivalent if there is a diagram



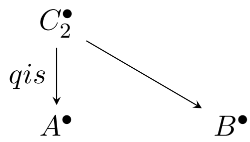
which commutes up to homotopy. In particular

$$(C^\bullet \to C_1^\bullet \to A^\bullet) \sim (C^\bullet \to C_2^\bullet \to A^\bullet).$$

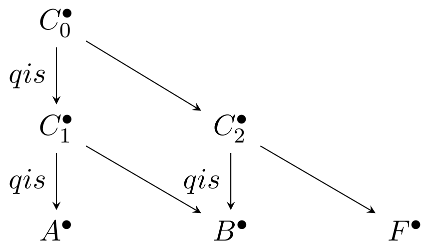
Morphisms



and



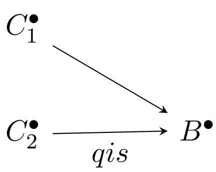
compose to give a diagram



which commutes up to homotopy.

Corollary 3.1. *The equivalence and the compositions exist and are well-defined.*

Proof: Existence of compositions comes from applying Proposition 2.4 to the diagram



Proposition 3.2. *The derived category $D(\mathcal{A})$ is defined by the following universal property: There exists a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ such that for a morphism $f : A^\bullet \to B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$, $Q(f)$ is an isomorphism whenever f is a quasi-isomorphism; for any functor $F : \mathrm{Kom}(\mathcal{A}) \to D$ satisfying this property there exists a unique functor $G : D(\mathcal{A}) \to D$ with $F \cong G \circ Q$.*

Example 3.3. Let $\mathcal{A} = Vec_f(k)$, the category of finite-dimensional vector spaces over a field k. Then $A^\bullet \in D(\mathcal{A})$ satisfies $A^\bullet = \bigoplus H^i(A^\bullet)[-i]$, and so we have $D(\mathcal{A}) \cong \prod_{i\in\mathbb{Z}} \mathcal{A}$.

Proposition 3.4. *Let $\mathcal{A}$ be an abelian category with enough injectives, and $\mathcal{I} \subset \mathcal{A}$ the full subcategory of injectives. Then the natural functor $\iota : K^+(\mathcal{I}) \to D^+(\mathcal{A})$, where the $+$ superscript denotes the subcategory of complexes which ae bounded below, is an equivalence.*