The Fourier-Mukai transform

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Everything here comes from Huybrechts's book [1] pages 86 and 113-122.

1 Definition and examples

We will have the following conventions: Let X and Y be smooth projective varieties over a field. We have the projections

$$p : X \times Y \to Y, \qquad q : X \times Y \to X.$$

We will not write the L's and R's in front of the functors but all functors we consider are in fact derived functors.

Definition 1.1. Let $P \in D^b(X \times Y)$, the induced Fourier-Mukai transform is the functor

$$\begin{aligned} \Phi_P \ : \ D^b(X) &\to D^b(Y) \\ \mathcal{E}^\bullet &\mapsto p_*(q^*(\mathcal{E}^\bullet) \otimes P). \end{aligned}$$

We say P is the Fourier-Mukai kernel of Φ_P.

Remark 1.2. Note that since q is flat, the derived functor q^* is just the usual pullback.

To be less ambiguous we could write $\Phi_P^{X \to Y}$ for the Fourier-Mukai transform defined above. We then also get a Fourier-Mukai transform $\Phi_P^{Y \to X} : D^b(Y) \to D^b(X)$ by reversing the roles of p and q in the definition. So one Fourier-Mukai kernel induces two Fourier-Mukai transforms. Unless we specify otherwise we take Φ_P to be the one from $D^b(X)$ to $D^b(Y)$.

Remark 1.3. The Fourier-Mukai Transform is a composition of three exact (i.e. triangulated) functors and is therefore itself exact (triangulated).

Now we will show some examples of functors that are in fact Fourier-Mukai transforms. We will use the projection formula that we saw before

$$f_*\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet \cong f_*(\mathcal{E}^\bullet \otimes f^*\mathcal{F}^\bullet). \tag{1}$$

Example 1.4. The identity

$$\mathrm{id} : D^b X \to D^b X$$

is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$, where Δ is the diagonal in $X \times X$. When we look at the diagonal embedding $i : X \xrightarrow{\sim} \Delta \subset X \times X$ we have $i_* \mathcal{O}_X = \mathcal{O}_\Delta$. We use this and the projection formula (1) to get

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(\mathcal{E}^\bullet) &= p_*(q^*\mathcal{E}^\bullet \otimes i_*\mathcal{O}_X) \\ &= p_*(i_*(i^*q^*\mathcal{E}^\bullet \otimes \mathcal{O}_X)) \\ &= (p \circ i)_*((q \circ i)^*\mathcal{E}^\bullet \otimes \mathcal{O}_X) \\ &= \mathcal{E}^\bullet\end{aligned}$$

Example 1.5. For a function $X \to Y$ we have the graph $X \xrightarrow{\Gamma_f} X \times Y$ where $\Gamma_f = \mathrm{id} \times f$. We have $\Gamma_{f*}\mathcal{O}_X = \mathcal{O}_{\Gamma_f}$ so similar to the identity case we get

$$\Phi_{\mathcal{O}_{\Gamma_f}}(\mathcal{E}^\bullet) = (p \circ \Gamma_f)_*((q \circ \Gamma_f)^*\mathcal{E}^\bullet \otimes \mathcal{O}_X) = f_*\mathcal{E}^\bullet.$$

We can reverse the roles of p and q to get

$$\Phi^{X \to Y}_{\mathcal{O}_{\Gamma_f}} = f_* \quad , \qquad \Phi^{Y \to X}_{\mathcal{O}_{\Gamma_f}} = f^*.$$

Taking global sections can be seen as a special case of this since for $f : X \to \operatorname{Spec} k$ we have $f_* = \Gamma$.

Example 1.6. If we were to take the diagonal embedding of a line bundle L on X rather than taking the whole diagonal, we get $\Phi_{i_*L}(\mathcal{E}^\bullet) = \mathcal{E}^\bullet \otimes L$.

Example 1.7. Taking the shift of the diagonal gives the shift, we have $\Phi_{\mathcal{O}_\Delta[1]}(\mathcal{E}^\bullet) = \mathcal{E}^\bullet \otimes \mathcal{O}_X[1] = \mathcal{E}^\bullet[1]$.

2 Adjoints and composition

We can express adjoints of the Fourier-Mukai transform in terms of its kernel. For this we need Grothendieck-Verdier duality. Let $f : X \to Y$, we define $\omega_f := \omega_X \otimes f^*\omega_Y^\vee$ and $\dim f := \dim X - \dim Y$.

Theorem 2.1 (Grothendieck-Verdier duality)**.** *Let $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{E}^\bullet \in D^b(Y)$, there is a functorial isomorphism*

$$f_*\mathscr{H}om(\mathcal{F}^\bullet, f^*\mathcal{E}^\bullet \otimes \omega_f[\dim f]) \cong \mathscr{H}om(f_*\mathcal{F}^\bullet, \mathcal{E}^\bullet).$$

Keep in mind that (as everywhere) the operations here are all derived functors.

We are interested in the special case where $f = q$ and we then take global sections. We get $\omega_f = \omega_{X \times Y} \otimes q^*\omega_X^\vee = p^*\omega_Y$ and

$$\mathrm{Hom}_{D^b(X \times Y)}(\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]) \cong \mathrm{Hom}_{D^b(X)}(q_*\mathcal{F}^\bullet, \mathcal{E}^\bullet). \qquad (2)$$

Definition 2.2. Let $P \in D^b(X \times Y)$, we define $P_L, P_R \in D^b(X \times Y)$

$$P_L = P^\vee \otimes p^*\omega_Y[\dim Y] \;, \qquad P_R = P^\vee \otimes q^*\omega_X[\dim X].$$

Let $\Phi_{P_L}, \Phi_{P_R} : D^b(Y) \to D^b(X)$ be the corresponding Fourier-Mukai transforms.

Proposition 2.3. *The Fourier-Mukai transforms $\Phi_{P_L}, \Phi_{P_R} : D^b(Y) \to D^b(X)$ are left, respectively right adjoint to Φ_P.*

Proof. We only proof it for Φ_{P_L}. We will use (2) and the fact that pullback and pushforward are adjoint.

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(\Phi_{P_L}(\mathcal{F}^\bullet), \mathcal{E}^\bullet) &= \mathrm{Hom}_{D^b(X)}(q_*(p^*\mathcal{F}^\bullet \otimes P_L), \mathcal{E}^\bullet) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(p^*\mathcal{F}^\bullet \otimes P_L, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(p^*\mathcal{F}^\bullet \otimes P^\vee, q^*\mathcal{E}^\bullet) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes P) \\
&= \mathrm{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, p_*(q^*\mathcal{E}^\bullet \otimes P)) \\
&= \mathrm{Hom}_{D^b(Y)}(\mathcal{F}^\bullet, \Phi_P(\mathcal{E}^\bullet))
\end{aligned}$$

□

Let $\pi_{XY} : X \times Y \times Z \to X \times Y$ be the projection, similarly we also have π_{XZ} and π_{YZ}. If we have $P \in D^b(X \times Y)$ and $Q \in D^b(Y \times Z)$ we define

$$R := \pi_{XZ*}({\pi_{XY}}^*P \otimes {\pi_{YZ}}^*Q) \in D^b(X \times Z).$$

Proposition 2.4. *The diagram*

$$D^b(X) \xrightarrow{\Phi_P} D^b(Y) \xrightarrow{\Phi_Q} D^b(Z), \qquad D^b(X) \xrightarrow{\Phi_R} D^b(Z)$$

commutes.

3 Orlov's theorem

Theorem 3.1 (Orlov)**.** *Let F be a fully faithful functor*

$$F : D^b(X) \to D^b(Y)$$

that admits a left and a right adjoint. There exists a $P \in D^b(X \times Y)$, unique up to unique isomorphism, such that $\Phi_P \cong F$.

It turns out that the condition on being fully faithful can be weakened to something less than full. And the condition on the existence of adjoints can even be dropped altogether.

Corollary 3.2. *If there exists an equivalence of categories* $D^b(X) \to D^bY$ *then* $\dim X = \dim Y$.

Proof. By Orlov's theorem there exists a P such that Φ_P is the equivalence. The adjoints Φ_{P_L}, Φ_{P_R} are then the quasi-inverses of Φ_P. This means that $\Phi_{P_L} \cong \Phi_{P_R}$. Now we can use Orlov's theorem again to see that the kernel must be unique so $P_L \cong P_R$. When we write this out we get

$$P^\vee \cong P^\vee \otimes q^*\omega_X \otimes p^*\omega_Y^\vee[\dim X - \dim Y].$$

Because $P^\vee$ is a bounded complex, there can be no isomorphism if a shift occurs on the right hand side. Because p and q are flat $q^*\omega_X \otimes p^*\omega_Y^\vee$ is concentrated in degree zero and no shift occurs there. This means that for there to not be a shift we need $\dim X - \dim Y = 0$. □

Whenever we have a morphism $\psi : P \to Q$ of objects $P, Q \in D^b(X \times Y)$ we get a corresponding morphism Φ_ψ. This gives us a functor

$$\begin{aligned} \Phi : D^b(X \times Y) &\to D^b(Y)^{D^b(X)} \\ P &\mapsto \Phi_P \\ \psi &\mapsto \Phi_\psi \end{aligned}$$

Here $D^b(Y)^{D^b(X)}$ is the category of functors from $D^b(X)$ to $D^b(Y)$.

We show in the following example that this functor is not faithful.

Example 3.3. Let E be an elliptic curve and consider the diagonal Δ inside $E \times E$. Using Serre duality we have

$$\mathrm{Ext}^2_{D^b(E\times E)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathrm{Ext}^0_{D^b(E\times E)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes \omega_{E\times E}) \neq 0.$$

So there is a morphism $\psi : \mathcal{O}_\Delta \to \mathcal{O}_\Delta[2]$ that is not the zero morphism. We get a corresponding $\Phi_\psi : \Phi_{\mathcal{O}_\Delta} \to \Phi_{\mathcal{O}_\Delta[2]}$ which as we saw in our earlier examples is a map $\Phi_\psi : \mathrm{id} \to [2]$.

If this map is to be nonzero then there must be a nonzero function in $\mathrm{Ext}^2_{D^b(E)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet)$ for some $\mathcal{F}^\bullet \in D^b(E)$. When we look at a complex of sheaves concentrated in degree zero then by Serre duality we have $\mathrm{Ext}^2_{D^b(E)}(\mathcal{F}, \mathcal{F}) = 0$, since $2 > \dim E$. We now use the fact that for curves any complex of sheaves $\mathcal{F}^\bullet$ can be written as a sum of complexes of sheaves concentrated in a single degree, i.e. $\mathcal{F}^\bullet \cong \bigoplus \mathcal{F}_i[i]$. So we find that $\mathrm{Ext}^2_{D^b(E)}(\mathcal{F}^\bullet, \mathcal{F}^\bullet) = 0$ and therefore $\Phi_\psi = 0$.

References

[1] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006.