Derived Functors

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Contents

1 Introduction

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. We have seen that it induces a functor between the homotopy categories. In particular under the hypothesis that F is exact we have a commutative diagram as follows

$$\begin{tikzcd} \mathrm{K}(\mathcal{A}) \arrow[r] \arrow[d, "Q_{\mathcal{A}}"'] & \mathrm{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathrm{D}(\mathcal{A}) \arrow[r, dashed] & \mathrm{D}(\mathcal{B}) \end{tikzcd} \tag{1}$$

Hence, the functor F also descends at the level of derived categories.

Remark 1.1. *Indeed, if F is exact we have that*

i) F sends quasi-isomorphisms to quasi-isomorphisms: this follows from the fact that, if $A^\bullet \to B^\bullet \to C^\bullet$ is a triangle in $K(\mathcal{A})$ with $A^\bullet \xrightarrow{qis} B^\bullet$, then $C^\bullet$ is acyclic, that is $H^i(C^\bullet) = 0\ \forall i$. Then, if we apply F we get a triangle $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ where $F(C^\bullet)$ is again acyclic (since F and H^i commute) and so $F(A^\bullet) \xrightarrow{qis} F(B^\bullet)$.

ii) F sends acyclic objects to acyclic objects: if $A^\bullet \to B^\bullet \to C^\bullet$ is a triangle in $K(\mathcal{A})$ with $C^\bullet$ acyclic, then $A^\bullet \to B^\bullet$ is a quasi-isomorphism so $F(A^\bullet) \to F(B^\bullet)$ is also a quasi-isomorphism, hence $F(C^\bullet)$ is acyclic.

Remark 1.2. *The above construction rely on the hypothesis that F is exact. If F is not exact, the situation described in (1) does not hold anymore. Indeed, let F be an additive left (or right) exact functor and consider an acyclic complex $X^\bullet$. Then we have that $X^\bullet \to 0$ is a quasi-isomorphism, hence when we apply F we have that*

$$F(X^\bullet) \xrightarrow{F(0)=0} 0$$

is not always a quasi-isomorphism because $F(X^\bullet)$ might not be acyclic. Therefore, F does not send quasi-isomorphisms to quasi-isomorphisms.

Example 1.3. *An instance of the situation described in Remark 1.2 is the following. Let us consider the abelian category $\mathcal{A} = \mathbb{Z}$-mod, then the complex*

$$X^\bullet = 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

is acyclic (since $H^i(X^\bullet) = 0\ \forall i$), hence in particular $X^\bullet \to 0$ is a quasi-isomorphism. However, if we apply the (right exact) functor

$$F = - \otimes \mathbb{Z}/p\mathbb{Z}$$

to the previous complex, we get

$$F(X^\bullet) = \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \to 0$$

which is not acyclic and so in particular $F(X^\bullet) \xrightarrow{0} 0$ is not a quasi-isomorphism.

In any setting, one works with functors which might or might not be exact. For example, let X, Y be topological spaces and $f : X \to Y$ a continuous function, then at the level of sheaves we have that

the inverse image $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is exact
the direct image $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is left exact
the global sections $\Gamma(X, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$ is left exact
the Hom-functor $\mathrm{Hom}(A, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$ is left exact
the tensor product $A \otimes - : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$ is right exact

Problem: We need to find a procedure to induce a functor at the level of derived categories starting from $F : \mathcal{A} \to \mathcal{B}$ between abelian categories without assuming the exactness of F.

Remark 1.4. *Note that Remark 1.2 shows that applying degree-wise a functor that is not exact will not solve the above problem.*

2 Injective Objects

Definition 2.1. *Let $\mathcal{A}$ be an abelian category, $I \in \mathrm{Ob}(\mathcal{A})$ is an* **injective object** *if for any monomorphism $Y \hookrightarrow X$ and morphism $Y \to I$ in $\mathcal{A}$ there exists a morphism $\phi : X \to I$ in $\mathcal{A}$ such that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d] & X \arrow[dl, dashed, "\exists\phi"] \\ & I & \end{tikzcd}$$

commutes.

Remark 2.2. *Note that $I \in \mathrm{Ob}(A)$ is injective if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact.*

Definition 2.3. *Let $\mathcal{A}$ be an abelian category and consider an $X^\bullet \in \mathrm{Ob}(Kom^+(\mathcal{A}))$, then $I^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ is an* **injective resolution of** $X^\bullet$ *if I^i is injective $\forall i$ and there exists a quasi-isomorphism $s : X^\bullet \to I^\bullet$.*

Example 2.4. *An injective resolution of an object $A \in \mathrm{Ob}(\mathcal{A})$, that is a complex concentrated in a single degree, is an exact sequence of the form*

$$0 \to A \to I^0 \to I^1 \to \dots$$

Definition 2.5. *We say that an abelian category $\mathcal{A}$* **has enough injectives** *if any $A \in \mathrm{Ob}(\mathcal{A})$ is such that there exists $A \hookrightarrow I$ with I injective.*

Proposition 2.6 (Existence of Injective Resolutions)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives, then $\forall A^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ there exists $I^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ injective resolution of $A^\bullet$ with $A^\bullet \to I^\bullet$ a quasi-isomorphism.*

Proof. See [Huy06, Proposition 2.35] or [Ive12, III.5]. □

Lemma 2.7. *Let $\mathcal{A}$ be an abelian category with enough injectives and $A^\bullet, I^\bullet \in \mathrm{Ob}(Kom^+(\mathcal{A}))$ with $I^\bullet$ injective, then*

$$\mathrm{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, \mathcal{I}^\bullet)$$

Proof. See [Huy06, Lemma 2.39]. □

Remark 2.8. *Lemma 2.7 is saying that we con consider morphisms between a complex and an injective complex in the homotopy category rather than in the derived category. Note that this simplify a lot the situation since one does not need to deal with the calculus of fractions explained in the previous talk.*

Definition 2.9. *Let us denote by* $\mathcal{I} \subset \mathcal{A}$ *the* **full abelian sub-category of injective objects** *of* $\mathcal{A}$.

Remark 2.10. *Note that* $K^+(\mathcal{I}) \hookrightarrow K^+(\mathcal{A})$ *inherits the structure of triangulated category.*

Proposition 2.11. *Let* $\mathcal{A}$ *be an abelian category with enough injectives, then the functor*

$$i : K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is an equivalence of categories.

Proof. It is enough to show that the above functor is fully faithful and essentially surjective. See [Huy06, Proposition 2.40]. □

3 Derived Functors and Their Properties

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories where in particular $\mathcal{A}$ has enough injectives. Using the previous results, we can consider the diagram

$$\begin{tikzcd}
\mathrm{K}^+(\mathcal{I}) \arrow[r, hook] \arrow[dr, "i"] & \mathrm{K}^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"] & \mathrm{K}^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
& \mathrm{D}^+(\mathcal{A}) \arrow[ul, dashed, bend left, "i^{-1}"] & \mathrm{D}^+(\mathcal{B})
\end{tikzcd}$$

Definition 3.1. *The* **right derived functor of F** *is given by*

$$RF := Q_{\mathcal{B}} \circ K(F) \circ i^{-1}.$$

Remark 3.2. *Note that after the restriction to* $K^+(\mathcal{I})$ *quasi-isomorphisms are sent to quasi-isomorphisms and so the restriction extends to a functor*

$$D^+(\mathcal{A}) \cong K^+(\mathcal{I}) \cong D^+(\mathcal{I}) \to D^+(\mathcal{B}).$$

Definition 3.3. *The* **higher derived functors of F** *are given by*

$$R^i F := H^i \circ RF.$$

Remark 3.4. *Let us list some properties of the (higher) derived functors*

1. RF *is exact since it is composition of exact functors.*
2. $R^i F = 0$ *if* $i < 0$.
3. $R^0 F(A) = F(A)$ *because*

   $$R^i F(A) = H^i(\ldots \to F(I^0) \xrightarrow{\alpha} F(I^1) \to \ldots)$$

   hence

   $$R^0 F(A) = \ker \alpha = F(A)$$

   since F *is left exact*
4. *Given a short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$, *we have that* $R^i F$ *induces a long exact sequence of the form*

   $$0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to R^1 F(B) \to R^1 F(C) \to R^2 F(A) \to \ldots$$

   This follows from the fact that such short exact sequence in $\mathcal{A}$ *corresponds to a distinguished triangle* $A \to B \to C \to A[1]$ *in* $D^+(\mathcal{A})$ *and, after applying* RF *to such triangle, one gets a distinguished triangle* $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ *in* $D^+(\mathcal{B})$. *Considering* H^i *of the latter triangle gives the claimed long exact sequence.*
5. *The same procedure explained for a left exact functor works for a right exact functor* $F : \mathcal{A} \to \mathcal{B}$ *under the assumption that* $\mathcal{A}$ *is an abelian category with enough projectives. Using projective objects and projective resolutions (instead of injective objects and injective resolutions) allows to construct the left derived functor* $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$.

6. *There is a more general setting one can consider in order to defined a derived functor, see [Huy06, p. 48] or [GM13, III.6.3]*

   Definition 3.5. *A triangulated sub-category $\mathcal{K} \subset K^+(\mathcal{A})$ is* **adapted to** *F if*

   i) given $A^\bullet \in \mathcal{K}$ acyclic then $F(A^\bullet)$ is acyclic

   ii) any $A^\bullet \in K^+(\mathcal{A})$ is quasi-isomorphic to a complex in $\mathcal{K}$.

   In particular, using adapted classes allows to avoid projective resolution (that in some cases do not exist) and injective resolutions (which can be quite difficult to handle) considering instead other resolutions (for example free, flat, flabby, etc.).

7. *If $F_1 : \mathcal{A} \to \mathcal{B}$ and $F_2 : \mathcal{B} \to \mathcal{C}$ are left exact functors between abelian categories such that there exist adapted classes $\mathcal{I}_1 \subset \mathcal{A}$ and $\mathcal{I}_2 \subset \mathcal{B}$ with $F_1(\mathcal{I}_1) \subset \mathcal{I}_2$ then we have a natural isomorphism*
$$R(F_2 \circ F_1) \cong RF_2 \circ RF_1.$$
   For instance, see [Huy06, Proposition 2.58].

8. *Another advantage of considering R^iF is given by the fact that one gets a complex and not just some objects; in particular, this means that R^iF carries much more information.*

Example 3.6. *Let $\mathcal{A}$ be an abelian category with enough injectives and $A \in \mathrm{Ob}(\mathcal{A})$, then the functor*
$$\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \to Ab$$
is left exact. Its higher derived functors are given by
$$\mathrm{Ext}^i_{\mathcal{A}}(A, -) := H^i \circ R\,\mathrm{Hom}_{\mathcal{A}}(A, -).$$
In particular, if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\mathcal{A}$, applying the functor $\mathrm{Ext}^i_{\mathcal{A}}(A, -)$ yields the well known long exact sequence
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,X) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,Y) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A,Z) \arrow[dll] \\ & \mathrm{Ext}^1_{\mathcal{A}}(A,X) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}}(A,Y) \arrow[r] & \dots \end{tikzcd}$$

Example 3.7. *Let us go back to Example 1.3 and consider again the acyclic complex*
$$X^\bullet = 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$
in the category $\mathcal{A} = \mathbb{Z}$-mod. We saw that applying $F = - \otimes \mathbb{Z}/p\mathbb{Z}$ does not give an acyclic complex because F is only right exact. Now, if we apply the higher left derived functor L^iF to $X^\bullet$ we get
$$L^iF(X^\bullet) = 0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z} \to 0.$$
Indeed, the first term that appears after the zero on the left in $L^iF(X^\bullet)$ is
$$L^1F(\mathbb{Z}/p\mathbb{Z}) \cong L^1F(P(\mathbb{Z}/p\mathbb{Z})) \cong H^1(F(\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z})) \cong H^1(\mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$
where $P(\mathbb{Z}/p\mathbb{Z})$ denotes a projective resolution of $\mathbb{Z}/p\mathbb{Z}$, that is $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$. That this term can be identified with $Tor_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$. Moreover, the complex $L^iF(X^\bullet)$ is now acyclic.

Proposition 3.8. *Let $\mathcal{A}$ be an abelian category with enough injectives and consider $A, B \in \mathrm{Ob}(\mathcal{A})$, that is A and B are two complexes concentrated in degree zero, then*
$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[i]).$$

Proof. See [Huy06, Proposition 2.56]. □

Remark 3.9. *In particular, note that if $i < 0$ Proposition 3.8 implies that there are no morphism in the derived category between A and $B[i]$.*

References

[GM13] S.I. Gelfand and Y.J. Manin. *Methods of Homological Algebra.* Springer Berlin Heidelberg, 2013.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Oxford University Press on Demand, 2006.

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