Fig. 3. Coupled microstrip.

TABLE I

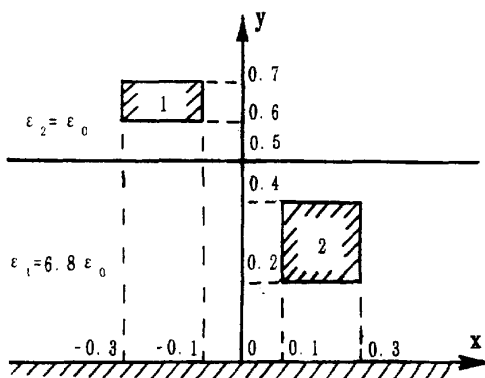| capacitance F/m	this method	reference [1]	reference [2]
C (1, 1)	0. 9213E-10	0. 9165E-10	0. 9236e-10
C (1, 2)	-0. 8302E-11	-0. 8220E-11	-0. 8494E-11
C (2, 1)	-0. 8302E-11	-0. 8220e-11	-0. 8494E-11
C (2, 2)	0.·9213E-10	0. 9165E-10	0. 9236E-10

Fig. 4. Two conductors in two different dielectric layers.

TABLE II

capacitance F/m	this method	reference [1]	reference [4]
C (1, 1)	0. 3697E-10	0. 3651E-10	0. 3701E-10
C (1, 2)	-0. 1584E-11	-0. 1562E-11	-0. 1520E-11
C (2, 1)	-0. 1584E-11	-0. 1562E-11	-0. 1520E-11
C (2, 2)	0. 2134E-9	0. 2099E-10	0. 2108E-9

Example 2: There are two different rectangular conductors in two dielectric layers above a ground plane as shown in Fig. 4. The results using this method together with those of [1] and [4] are shown in Table II, and the computing speed of this method is also much faster than those of other methods.

IV. CONCLUSION

A new method for calculating the capacitance matrix of the multiconductor interconnects is given. The computing speed is faster than that of other methods with the same accuracy, and the desired storage of the computer is also decreased, so this method is effective for computing the electrical parameters of the interconnects for high-speed/high-complexity electronic systems.

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Eigenmode Sequence for an Elliptical Waveguide with Arbitrary Ellipticity

Shan-jie Zhang and Yao-chun Shen

Abstract—**Eigenmode sequence for an elliptical waveguide with arbitrary ellipticity is studied by directly calculating the parametric zeros of the modified Mathieu functions of the first kind and their derivatives. The normalized cutoff wavelength of the lowest 100 successive modes are presented, and the curvefitting expressions for the determination of the cutoff wavelength of the lowest 10 order modes are given, which are valid for the ellipticities ranging from 0.0 to 0.99.**

I. INTRODUCTION

Elliptical waveguides have wide applications such as radar feed lines, multichannel communication and accelerator beam tubes. The determination of the cutoff wavelength of the elliptical waveguide is one of the most important problems for designing the waveguide or analyzing the wave propagation in the waveguide. In 1938, Chu [1] first presented the theory of the transmission of the electromagnetic waves in elliptical waveguide. Since then some more numerical results about the cutoff wavelengths in elliptical waveguide have been obtained [2]–[4]. In 1970, Kretzschmar [5] obtained the curves of the cutoff wavelengths for the 19 successive modes and the approximative formula for the eight lowest order modes. Recently Goldberg [6] calculated the cutoff wavelengths for the six lowest modes and gave a correction to the field pattern plotted in [1]. In fact, the determination of the cutoff wavelength of an elliptical waveguide is a problem of calculating the zeros of the modified

TABLE I
NORMALIZED CUTOFF WAVELENGTHS FOR AN ELLIPTIC WAVEGUIDE λ_c/a

No.	Eccentricity $e = 0.1$		Eccentricity $e = 0.5$		Eccentricity $e = 0.9$	
	Mode	λ_c/a	Mode	λ_c/a	Mode	λ_c/a
1	TEc1–1	3.41188201	TEc1–1	3.39447796	TEc1–1	3.34819934
2	TEs1–1	3.39617177	TEs1–1	2.97448032	TEc2–1	1.82871198
3	TMc0–1	2.60616852	TMc0–1	2.41960702	TEs1–1	1.56495481
4	TEc2–1	2.05208977	TEc2–1	1.94968828	TMc0–1	1.49062515
5	TEs2–1	2.05203202	TEs2–1	1.90795125	TEc3–1	1.26542321
6	TMc1–1	1.63772228	TMc1–1	1.57615210	TEs2–1	1.22921614
7	TEc0–1	1.63563513	TEc0–1	1.49938950	TMc1–1	1.16070249
8	TMs1–1	1.63361256	TMs1–1	1.46727452	TEs3–1	0.99860472
9	TEc3–1	1.49183132	TEc3–1	1.40275680	TEc4–1	0.96975846
10	TEs3–1	1.49183109	TEs3–1	1.39790776	TMc2–1	0.93753838
11	TMc2–1	1.22042951	TMc2–1	1.16520501	TEs4–1	0.83397716
12	TMs2–1	1.22037489	TMs2–1	1.13357998	TEc0–1	0.81765942
13	TEc4–1	1.17863600	TEc1–2	1.12835796	TMs1–1	0.80930877
14	TEs4–1	1.17863600	TEc4–1	1.10587635	TEc5–1	0.78724715
15	TEc1–2	1.17713084	TEs4–1	1.10529053	TMc3–1	0.78026278
16	TEs1–2	1.17396194	TEs1–2	1.04785956	TEc1–2	0.71603488
17	TMc0–2	1.13533477	TMc0–2	1.03034970	TEs5–1	0.71232947
18	TMc3–1	0.98234023	TMc3–1	0.92887494	TMs2–1	0.70828320
19	TMs3–1	0.98233977	TMs3–1	0.92078076	TMc4–1	0.66513122
20	TEc5–1	0.97690644	TEc5–1	0.91607169	TEc6–1	0.66328788
21	TEs5–1	0.97690644	TEs5–1	0.91599927	TEc2–2	0.63319257
22	TEc2–2	0.93464260	TEc2–2	0.89716685	TMs3–1	0.62616630
23	TEs2–2	0.93456961	TEs2–2	0.86420375	TEs6–1	0.61969135
24	TMc1–2	0.89446309	TMc1–2	0.84778529	TMc5–1	0.57800441
25	TEc0–2	0.89328695	TEc0–2	0.80277908	TEc7–1	0.57358748
30	TMs4–1	0.82593477	TMs4–1	0.77560113	TEs7–1	0.54728403
40	TMc5–1	0.71452786	TEs1–3	0.65137311	TEc5–2	0.46043562
50	TEc2–3	0.62874749	TMs6–1	0.59214061	TMs7–1	0.41637935
60	TMs4–2	0.56644221	TMs4–2	0.53235279	TMc9–1	0.37563007
70	TEs6–2	0.53408707	TEc1–4	0.49402711	TEc12–1	0.34500891
80	TEc11–1	0.48863340	TMc3–3	0.46038334	TEc4–3	0.32458532
90	TMs1–4	0.46977544	TMs6–2	0.43415539	TEc5–3	0.30594802
100	TEs8–2	0.44401270	TEc0–4	0.41616560	TMs12–1	0.28669920

Mathieu functions of the first kind, i.e., $Se_m(\xi, q)$ and $Ce_m(\xi, q)$, and their derivatives, where the two separate parameters ξ and q, as will be discussed in the following section, are related to the dimension size and cutoff wavelength of the elliptical waveguide respectively. In most of the previous work, the cutoff wavelength were determined by calculating the zeros ξ for a given q since it is much easier to determine the zeros ξ than the parametric zeros q of the functions. However, it is not convenient to determine the eigenmode sequence for an elliptical waveguide with given ellipticity since a large number of calculations are required. Furthermore, it may also cause omission of the high order modes in eigenmode sequence since the succession of the various modes is a function of the ellipticity. Thus we need a more direct and convenient way to determine the eigenmode sequence for a given elliptical waveguide.

In this paper, the cutoff wavelength sequence is determined by directly calculating the parametric zeros q of the modified Mathieu functions of the first kind and their derivatives. The calculation are made on an IBM PC-386 using Bessel functions series. The first 100 successive modes are presented for eccentricities 0.1, 0.5 and 0.9. The curve fitting expressions for the determination of the cutoff wavelength of the 10 lowest order modes are given. The accuracy is 10^{-8} for main mode TE_{c11}, 3×10^{-4} for other modes.

II. OUTLINE OF THE THEORY

Electromagnetic waves propagating in the elliptical waveguide are the combination of the TM and TE waves. For TM waves, the longitudinal components of the waves are: $H_z = 0$, $E_z = \psi$; for TE waves, $E_z = 0$, $H_z = \psi$. Where ψ is the general solution of the following wave equation in the orthogonal elliptical coordinate system

$$\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + 2q(\cosh 2\xi - \cos 2\eta) \right] \psi = 0. \tag{1}$$

The separate parameter q is defined as

$$q = \frac{a^2 e^2}{4}(\omega^2 \epsilon \mu - \beta^2) \tag{2}$$

where a and e are the semi-major axis and ellipticity of the waveguide, ω and β are the wave frequency and phase constant, respectively. Using the method of separation of variables we can obtain the following solution for the wave equation

$$\psi(\xi, \eta, z) = \begin{bmatrix} ce_m(q, \eta) \\ se_m(q, \eta) \end{bmatrix} \begin{bmatrix} Ce_m(\xi, q) \\ Se_m(\xi, q) \end{bmatrix} e^{j(\omega t - \beta z)}. \tag{3}$$

In these equations, ce_m and se_m are ordinary even and odd Mathieu functions, while Ce_m and Se_m are corresponding modified Mathieu functions of the first kind and order m. We can see from (3) that the longitudinal components E_z and H_z have two different forms corresponding to even and odd modes. Hence, there are four different mode types in an elliptical waveguide, denoted as TM_{cm}, TM_{sm}, TE_{cm} and TE_{sm}. Where the first subscript c (cos-type) and s (sin-type) represent even and odd modes, while the second subscript m is related to the order of the modified Mathieu functions of the first kind.

The tangent components of the electric field, which can be obtained from longitudinal components by applying Maxwell's Equations [4], [5], should be zero on the wall $\xi = \xi_0$, where ξ_0 is the radial coordinate of the elliptical boundary. Thus the boundary conditions can be written as

$$\begin{array}{lll} Ce_m(\xi_0, q) = 0 & \text{for} & TM_{cm} \quad \text{mode} \\ Se_m(\xi_0, q) = 0 & \text{for} & TM_{sm} \quad \text{mode} \\ Ce'_m(\xi_0, q) = 0 & \text{for} & TE_{cm} \quad \text{mode} \\ Se'_m(\xi_0, q) = 0 & \text{for} & TE_{sm} \quad \text{mode} \end{array} \tag{4}$$

<div align="center">

TABLE II

CURVEFITTING FORMULAS FOR DETERMINING THE CUTOFF WAVELENGTHS

</div>

Mode	Formula	Interval of e
TE_{c11}:	$\lambda_c/a = 3.41257911 - .06960165e^2 - .010811761e^4 - .001551343e^6 - .000196037e^8$	[0, 1.0]
TE_{s11}:	$\lambda_c/a = 3.41257911 - 1.64946e^2 - .193153e^4 - 1.26437e^6 + 2.02088e^8 - 1.67104e^{10}$	[0, 0.9]
	$\lambda_c/a = 1.56495481 - 7.2118(1-e)^{0.125} + 9.5048(1-e)^{0.25} + .148137(1-e)^{0.375}$	[0.9, 0.99]
TM_{c01}:	$\lambda_c/a = 2.61274057 - .666902e^2 - .173694e^4 - 1.40916e^6 + 2.1687e^8 - 1.87123e^{10}$	[0, 0.9]
	$\lambda_c/a = 1.49062515 - 7.96272(1-e)^{0.125} + 12.7149(1-e)^{0.25} - 2.78747(1-e)^{0.375}$	[0.9, 0.99]
TE_{c21}:	$\lambda_c/a = 2.05720298 - .515017e^2 + .301521e^4 + .285292e^6 - .590669e^8 + .268172e^{10}$	[0, 1.0]
TE_{s21}:	$\lambda_c/a = 2.05720298 - .52526e^2 - .0980927e^4 - 1.08694e^6 + 1.72239e^8 - 1.43919e^{10}$	[0, 0.9]
	$\lambda_c/a = 1.22921614 - 7.2778(1-e)^{0.125} + 13.3726(1-e)^{0.25} - 4.88771(1-e)^{0.375}$	[0.9, 0.99]
TM_{c11}:	$\lambda_c/a = 1.63978796 - .218162e^2 + .058415e^4 - 1.2277e^6 + 2.12388e^8 - 1.7275e^{10}$	[0, 0.9]
	$\lambda_c/a = 1.16070249 - 7.91779(1-e)^{0.125} + 16.1757(1-e)^{0.25} - 7.48142(1-e)^{0.375}$	[0.9, 0.99]
TE_{c31}:	$\lambda_c/a = 1.63978796 - .415913e^2 - .426159e^4 - .961653e^6 + 1.64945e^8 - 1.1602e^{10}$	[0, 0.9]
	$\lambda_c/a = 0.81765942 - 3.69985(1-e)^{0.125} + 4.66541(1-e)^{0.25} + .35838(1-e)^{0.375}$	[0.9, 0.99]
TM_{s11}:	$\lambda_c/a = 1.63978796 - .621608e^2 - .150714e^4 - .693906e^6 + 1.05299e^8 - .896293e^{10}$	[0, 0.9]
	$\lambda_c/a = 0.80930877 - 3.77731(1-e)^{0.125} + 5.00489(1-e)^{0.25} + .0443174(1-e)^{0.375}$	[0.9, 0.99]
TE_{c41}:	$\lambda_c/a = 1.49557313 - .371402e^2 - .065832e^4 + .247781e^6 + .105781e^8 - .181746e^{10}$	[0, 1.0]
TE_{s31}:	$\lambda_c/a = 1.49557313 - .383519e^2 + .127985e^4 - .9166e^6 + 1.50186e^8 - 1.23193e^{10}$	[0, 0.9]
	$\lambda_c/a = 0.99860472 - 6.71145(1-e)^{0.125} + 14.0357(1-e)^{0.25} - 6.78048(1-e)^{0.375}$	[0.9, 0.99]

with $\cosh\xi_0 = 1/e$. As the parameter q is related to the cutoff wavelength by (2), and there are a series of q values satisfying above equations. To avoid ambiguity, a third subscript n, corresponding to the nth parametric root, is required in the mode designation. Thus the complete designation of the waves propagating in an elliptical waveguide is TM_{cmn}, TM_{smn}, TE_{cmn} and TE_{smn}. The normalized cutoff wavelength can be obtained from Eq. (2) as

$$\lambda_c/a = \frac{\pi e}{\sqrt{q_{mn}}} \tag{5}$$

q_{mn} is the nth parametric zero of the modified Mathieu functions of the first kind of the order m or their derivatives.

III. METHOD AND RESULTS

A. Eigenmode Sequence

As mentioned above, there are four types of eigenmode sequence in an elliptical waveguide. For a given type TM_{smn}, there is following relationship among the zeros of the modified Mathieu functions. The value of the $(n+1)$th zero is larger than that of nth zero of the modified Mathieu functions, i.e., $q_{m,n+1} > q_{m,n}$; the value of the first zero of the modified Mathieu function of order $(m+1)$ is larger than the zeros of modified Mathieu function of order m, i.e., $q_{m+1,1} > q_{m,1}$. Thus only one initial value is needed for getting a type of eigenmode sequence.

The other types of eigenmode sequence TM_{cmn}, TE_{cmn} and TE_{smn} can be obtained through similar process. The eigenmode sequence of the elliptical waveguide with a given ellipticity can finally be determined by comparison. It is obvious from preceding discussions that no high order modes in the eigenmode sequence will be omitted by using this method. It should be pointed out that: the value of the parameter q will vary from 10^{-3} to 10^3 when lowest 100 modes with different ellipticity are considered. Therefore, a combination of bi-section and Regula falsi methods together with step-variable search method is necessary in order to calculate effectively the zeros of the modified Mathieu functions and their derivatives.

B. Characteristic Value

It is clear from (4), (5) that the exact computation of the modified Mathieu functions forms the main difficulty in the study of elliptical waveguides. These functions can be expanded by hyperbolic functions, Bessel functions and Bessel function products. The modified Mathieu function is the solution of the modified Mathieu equation

$$y'' - (b - 2q\cosh 2\xi)y = 0 \tag{6}$$

if b equals to the characteristic value b_m, which can be obtained by matrix method or following continued fractional method [4]

$$b_{2k}^c = -\frac{2q^2}{4-b-}\frac{q^2}{16-b-}\cdots$$

$$b_{2k}^s = 4 - \frac{q^2}{16-b-}\frac{q^2}{36-b-}\cdots$$

$$b_{2k+1}^c = 1 + q - \frac{q^2}{9-b-}\frac{q^2}{25-b-}\cdots$$

$$b_{2k+1}^s = 1 - q - \frac{q^2}{9-b-}\frac{q^2}{25-b-}\cdots. \tag{7}$$

Once b_m has been determined, the expansion coefficients can be easily obtained from (6) [4] [5]. Equation 7 is only valid for lower order m and larger value q. The instability of the conventional continued fractional method comes from the fact that one of the denominators of the continued fraction tends to be zero when q approaches to some special values. In order to get the exact characteristic values for larger m and smaller q, a modified continued fractional method is suggested as follow

$$b_{2k+p} = (2k+p)^2$$
$$- \left(\frac{q^2}{(2k+p+2)^2-b-}\frac{q^2}{(2k+p+4)^2-b-}\cdots\right)$$
$$- \left(\frac{q^2}{(2k+p-2)^2-b-}\frac{q^2}{(2k+p-4)^2-b-}\cdots\right.$$
$$\left.\frac{q^2}{(4-p)^2-b-q^2/B}\right). \tag{8}$$

where

$$p = 0, \quad B = 4 - b \qquad \text{for} \quad b_{2k}^c$$
$$p = 0, \quad B = 4 - b + 2q^2/b \quad \text{for} \quad b_{2k}^s$$
$$p = 1, \quad B = 1 + q - b \qquad \text{for} \quad b_{2k+1}^c \tag{9}$$
$$p = 1, \quad B = 1 - q - b \qquad \text{for} \quad b_{2k+1}^s$$

The combination of (7) and (8) can provide exact characteristic values of the modified Mathieu functions with large range of m and q.

C. Numerical Results

As a check of this method, we calculated 200 successive modes for elliptical waveguides with different ellipticities. Table. I lists the lowest 100 successive modes with ellipticities $e = 0.1$, 0.5 and 0.9. It is obvious from Table. I that the eigenmode sequence is a function of ellipticity, i.e., elliptical waveguide with different ellipticity has different eigenmode sequence. However, the main mode of the waveguide is always TE_{c11}. The first high order mode is TE_{s11} when $e < 0.8546001$ while it becomes TE_{c21} when $e > 0.8546001$.

As a large number of numerical calculation are required to determine the cutoff wavelength for a given mode and ellipticity, we presented here the curvefitting expressions for the determination of the cutoff wavelength of the lowest 10 order modes. The formulas for the different modes and their ranges of validity are given in Table II. Compared with previous works [5], [7], the expressions presented here have higher accuracy and are valid for wider range of ellipticity.

IV. CONCLUSION

We can conclude from above discussion that: 1) the modified continued fractional method suggested in this paper is suitable to calculate the characteristic values of the modified Mathieu functions with arbitrary order m and value q. 2) directly calculating the parametric zeros of the modified Mathieu functions of the first kind and their derivatives is an effective and easy way to determine the cutoff wavelength for a given elliptical waveguide, and ensures no omission of high order modes in the eigenmode sequence. 3) The normalized cutoff wavelength for the lowest 100 successive modes are presented, and the curvefitting expressions for the determination of the cutoff wavelength of the lowest 10 order modes are given, which have higher accuracy than previous calculations and are valid for wider range of ellipticity.

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A New Electric Field Integral Equation for Heterogeneous Dielectric Bodies of Revolution

Mark S. Viola

Abstract—In this paper, a novel electric field integral equation (EFIE) is developed for rotationally-symmetric heterogeneous dielectric bodies. This EFIE has several attractive features. Firstly, the azimuthal field component has been eliminated in this formulation thereby reducing the number of scalar unknowns from three to two. Secondly, it is a pure-integral equation in which there are no terms involving derivatives of the field components. Finally, this description is devoid of any highly singular kernel which would require a principal-value evaluation of the associated integral. These attributes render this formulation advantageous for both computational and theoretical pursuits.

I. INTRODUCTION

Rigorous analysis of electromagnetic phenomena within heterogeneous dielectric regions commonly proceeds from an integral or integro-differential equation for the electric field [1]–[5]. Construction of such an EFIE relies upon the identification of an equivalent volume density of polarization current. Inherently, this formulation is a volume integral equation having three scalar unknowns. Thus, its solution potentially poses a computationally intensive problem. Additional complications arise when the EFIE is cast in the form involving the electric dyadic Green's function [6]–[9]. However, the presence of certain symmetries allows the formulation of an alternative integral equation that provides both computational and theoretical advantages.

In this paper, a novel electric field integral equation (EFIE) is developed for heterogeneous dielectric bodies of revolution. It is assumed that the permittivity profile is azimuthally invariant. By exploiting the prevailing symmetry, straightforward analysis yields an EFIE having several appealing attributes. Firstly, the azimuthal field component is eliminated from the formulation in favor of the remaining (transverse) components. This reduction in the number of scalar unknowns from three to two facilitates numerical solution via standard techniques (e.g., the method of moments). Secondly, it is a rigorous pure integral equation for the transverse field components as opposed to an integro-differential one; no terms involving derivatives of the field components appear. Finally, the singularities of the kernels within this formulation are sufficiently weak, avoiding the necessitation of a principal-value integral and the corresponding depolarizing dyadic [7].

Throughout this paper, it shall be assumed that all media are linear, isotropic and magnetically homogeneous. Furthermore, the time dependence is harmonic $(e^{j\omega t})$ and is suppressed.

II. VOLUME-SURFACE INTEGRAL EQUATION DESCRIPTION

Attention is focused on Fig. 1, which depicts a dielectric body of revolution immersed in a uniform surround. A coordinate system is established such that the z-axis coincides with the axis of revolution. Open domain V, having boundary surface S with outer unit normal $\hat{n}$, is the region for the dielectric and is electrically characterized through its permittivity profile $\epsilon(\vec{r})$. In order to provide a well-posed problem, it is assumed that the closed region $\overline{V}$ is regular and that ϵ is