Solutions to Exercises

(i) $$F = -kT \ln Z = -kT \ln\left(\sum_{\{\sigma\}} \exp\left(\frac{J}{kT} \sum_{\langle ij \rangle} \sigma_i \sigma_j + \frac{1}{kT} \sum_i h_i \sigma_i \right)\right)$$

$$M = -\frac{\partial F}{\partial h_i} = +kT \frac{1}{Z} \frac{\partial Z}{\partial h_i} = +\frac{kT}{kT} \frac{1}{Z} \sum_{\{\sigma_i\}} \sigma_i \exp\left(-\frac{H}{kT}\right)$$
$$= \langle \sigma_i \rangle$$

$$\Rightarrow -kT \frac{\partial^2 F}{\partial h_i \partial h_j} = kT^2 \frac{\partial}{\partial h_i} \left(\frac{1}{Z} \sum_{\{\sigma\}} \frac{\sigma_j}{kT} \exp\left(-\frac{H}{kT}\right)\right)$$

$$= kT^2 \left[-\frac{1}{Z^2} \left(\sum_{\{\sigma\}} \frac{\sigma_j}{kT} \exp\left(-\frac{H}{kT}\right)\right)\left(\sum_{\{\sigma\}} \frac{\sigma_i}{kT} \exp\left(-\frac{H}{kT}\right)\right) + \frac{1}{Z} \sum_{\{\sigma\}} \frac{\sigma_j}{kT} \frac{\sigma_i}{kT} \exp\left(\frac{-H}{kT}\right) \right]$$

$$= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

(ii) $$\sum_{\underline{x}\,\underline{y}} \langle \phi(\underline{x},0)\, \phi(\underline{y},t) \rangle = V_s \sum_{\underline{x}} \langle \phi(\underline{0},0)\, \phi(\underline{x},t) \rangle$$

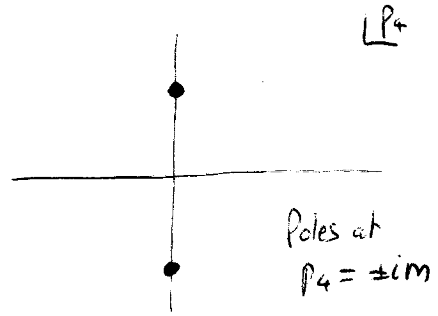
by translational invariance: V_s is spatial volume

$$= V_s \sum_{\underline{x}} \int_p \int_q \langle \phi(p) \phi(q) \rangle \, e^{-i\underline{q}\cdot\underline{x} - i q_4 t}$$

$$= V_s \sum_{\underline{x}} \int_p \frac{e^{i\underline{p}\cdot\underline{x} + i p_4 t}}{p^2 + m^2} \qquad \text{Now } \sum_{\underline{x}} e^{i\underline{p}\cdot\underline{x}} \propto \delta^3(\underline{p})$$

$\Rightarrow$ being cavalier about normalisation of F.T.:

$$\sum_{\underline{x}\,\underline{y}} \langle \phi(\underline{x},0)\, \phi(\underline{y},t) \rangle \propto \int dp_4 \frac{e^{i p_4 t}}{p_4^2 + m^2}$$



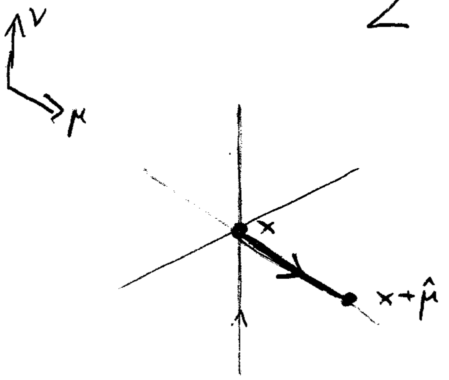
Complete contour in upper half-plane & use Jordan's lemma:

$$\Rightarrow \quad \propto \quad 2\pi i \cdot \frac{e^{-mt}}{2im}$$

ie timeslice propagator $\propto \frac{1}{m} e^{-mt}$

(ii)

$$\langle U_\mu(x) \rangle = \frac{1}{Z} \int DU \; U_\mu(x) \exp\left(\frac{\beta}{N} \sum_{x\mu<\nu} \operatorname{tr} \operatorname{Re} U_{\mu\nu}(x) \right)$$



Change variables on every other link emanating from site x

eg. $U_\nu(x) \mapsto U_\mu(x)\, U_\nu(x) \equiv U'_\nu(x)$
$U_\nu(x-\hat\nu) \mapsto U_\nu(x-\hat\nu)\, U_\mu^\dagger(x) \equiv U'_\nu(x-\hat\nu)$

$\Rightarrow$ all plaquettes containing link $x, x+\hat\mu$ become independent of $U_\mu(x)$

$$\operatorname{tr} U_{\mu\nu}(x) \mapsto \operatorname{tr} U'_{\mu\nu}(x) = \operatorname{tr}\{ U_\mu(x)\, U_\nu(x+\hat\mu)\, U_\mu^\dagger(x+\hat\nu)\, U_\nu^\dagger(x)\, U_\mu^\dagger(x) \}$$
$$= \operatorname{tr}\{ U_\nu(x+\hat\mu)\, U_\mu^\dagger(x+\hat\nu)\, U_\nu^\dagger(x) \}$$

Hence action is independent of $U_\mu(x)$
But also $dU' \equiv dU$ on transformed links, due to L & R invariance of Haar Measure

Hence $\langle U_\mu(x) \rangle = \frac{1}{Z} \int dU_\mu(x)\, U_\mu(x) \times \int D\tilde U \exp\left(S[\tilde U] \right)$

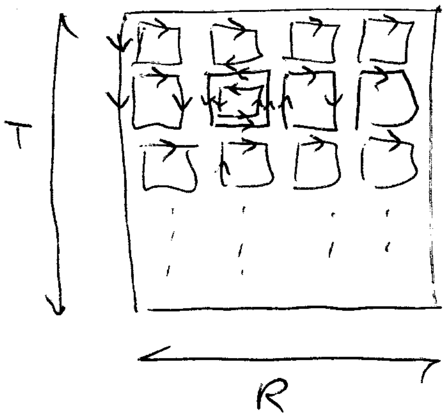
where $\tilde U$ does not depend on $U_\mu(x)$

But from integration rules $\int dU_\mu(x)\, U_\mu(x) = 0$

$$\Rightarrow \quad \langle U_\mu(x) \rangle \equiv 0$$

Solutions to Exercises

(iv) To achieve a tiling, need to expand exponential to $O\left(\frac{\beta}{2N}\right)^{RT+1}$



need to choose $(RT-1)$ terms from $\left(\sum U_{\mu\nu}\right)^{RT+1}$

" " " 2 " " $\left(\sum U^{+}_{\mu\nu}\right)^{RT+1}$

$$\Rightarrow \frac{1}{(RT+1)!} \quad \frac{(RT+1)!}{(RT-1)!\,2!} \quad (RT-1)!$$

from expansion of exp( )

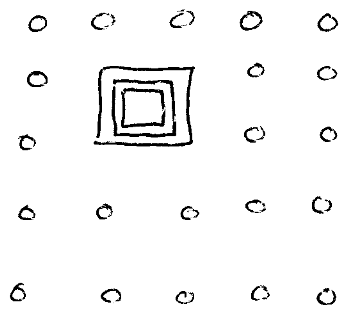
# of ways of choosing 2 from RT+1

# of indistinguishable tilings of "ordinary plaquettes"

But I can choose any one of the plaquettes to site

$$\Rightarrow \text{Overall numerical factor} = \frac{RT}{2}\left(\frac{\beta}{2N}\right)^{RT+1} \qquad (\text{Recall } N=3)$$

Now eliminate $\uparrow\!\downarrow$ as usual $\Rightarrow$ on $R(T+1)+T(R+1)-4$ links



$$\Rightarrow \left(\frac{1}{N}\right)^{2RT+T+R-4}$$

Similarly there are now $(R+1)(T+1)-4$ blobs $\circ$

$$\Rightarrow N^{RT+T+R+1-4}$$

Now, looks like

$$U_{i_1j_1}\,U_{j_1k_1}\,U_{k_1l_1}\,U_{l_1i_1}$$
$$U_{i_2j_2}\,U_{j_2k_2}\,U_{k_2l_2}\,U_{l_2i_2}$$
$$U_{i_3j_3}\,U_{j_3k_3}\,U_{k_3l_3}\,U_{l_3i_3}$$

use $\int dU\; U_{i_1j_1}U_{i_2j_2}U_{i_3j_3} = \frac{1}{3!}\,\varepsilon_{i_1i_2i_3}\,\varepsilon_{j_1j_2j_3}$

$$\Rightarrow \frac{1}{(3!)^4}\,\varepsilon_{i_1i_2i_3}\,\varepsilon_{j_1j_2j_3}\,\varepsilon_{j_1j_2j_3}\,\varepsilon_{k_1k_2k_3}\,\varepsilon_{k_1k_2k_3}\,\varepsilon_{l_1l_2l_3}\,\varepsilon_{l_1l_2l_3}\,\varepsilon_{i_1i_2i_3}$$

Finally note $\varepsilon_{i_1 i_2 i_3}\varepsilon_{i_1 i_2 i_3} = 3!$

$$\Rightarrow \quad \boxed{\square} = 1$$

$$\therefore \quad \langle W \rangle = \left\langle \frac{1}{N} \operatorname{tr} P \prod_{l \in \Gamma} U_l \right\rangle$$

$$= l.o. + \frac{1}{N}\,\frac{RT}{2}\left(\frac{\beta}{2N}\right)^{RT+1}\left(\frac{1}{N}\right)^{2RT+T+R-4} N^{RT+T+R+1-4}$$

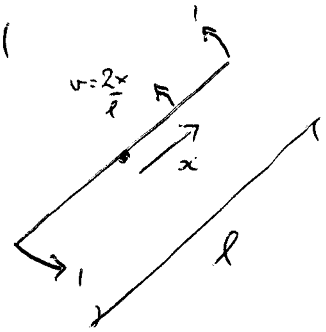
$$= l.o. + \left(\frac{\beta}{2N^2}\right)^{RT} \cdot \frac{RT\beta}{4N} = l.o. + \left(\frac{\beta}{18}\right)^{RT} \cdot \frac{RT\beta}{12} \quad \text{since } N=3$$

ie $\langle W \rangle \propto e^{-KRT} = \left(\frac{\beta}{18}\right)^{RT}\left(1 + \frac{RT\beta}{12} + O(\beta^2)\right)$

ie $Ka^2 = -\ln\left(\frac{\beta}{18}\right) - \frac{\beta}{12} + O(\beta^2)$

$\Rightarrow$ so next term reduces string tension, as string begins to fluctuate.

(v) With units $c = 1$



Mass of element of string between x, $x+dx$ $= \gamma_v K\,dx = \dfrac{K\,dx}{\sqrt{1-\frac{4x^2}{l^2}}}$

$$\Rightarrow \quad M = 2\int_0^{l/2} \frac{K\,dx}{\sqrt{1-\frac{4x^2}{l^2}}} = lK\int_0^{\pi/2} du = \boxed{\frac{\pi K l}{2} = M}$$

using $x = \frac{l}{2}\sin u$

angular momentum of element $= \gamma K x v\,dx = \dfrac{2Kx^2}{l\sqrt{1-\frac{4x^2}{l^2}}}$

$$\Rightarrow \quad J = \frac{4K}{l}\int_0^{l/2} \frac{x^2\,dx}{\sqrt{1-\frac{4x^2}{l^2}}} = \frac{Kl^2}{2}\int_0^{\pi/2} \sin^2 u\,du = \boxed{\frac{Kl^2}{8}\pi = J}$$

Eliminating l, we arrive at $J = \dfrac{1}{2\pi K} M^2$

Solution to Exercise

(vi)

To leading order in strong coupling $K = -\frac{1}{a^2} \ln\left(\frac{\beta}{2N^2}\right)$

$$\Rightarrow \quad a\frac{dK}{da} = \frac{2}{a^2}\ln\left(\frac{\beta}{2N^2}\right) - \frac{1}{\beta a^2}\, a\frac{\partial\beta}{\partial a} = 0$$ if K is "physical" (our assumption)

$$\Rightarrow \quad a\frac{\partial\beta}{\partial a} = 2\beta \ln\frac{\beta}{2N^2}$$

Now $B(g) = -a\frac{\partial g}{\partial a} \Rightarrow B(\beta) = -a\frac{\partial\beta}{\partial a}\cdot\frac{\partial g}{\partial\beta}$

$$\Rightarrow \quad B(\beta) = \sqrt{\frac{N}{2}}\cdot\frac{1}{\beta^{3/2}}\cdot 2\beta\ln\frac{\beta}{2N^2}$$

So for the equation which defines $\Delta\beta(\beta)$:

$$\int_a^{2a}\frac{da'}{a'} = \ln 2 = \sqrt{\frac{N}{2}}\int_\beta^{\beta-\Delta\beta}\frac{d\beta'}{\beta'^{3/2}B(\beta')} = \int_\beta^{\beta-\Delta\beta}\frac{d\beta'}{2\beta'\ln\left(\frac{\beta'}{2N^2}\right)}$$

Observe that $\frac{d}{dx}\ln\left(\ln\frac{x}{a}\right) = \frac{1}{x\ln\frac{x}{a}} \Rightarrow 2\ln 2 = \left[\ln\left(\ln\frac{\beta'}{2N^2}\right)\right]_\beta^{\beta-\Delta\beta}$

ie $$\ln\left(\frac{\beta-\Delta\beta}{2N^2}\right) = \ln\left(\left(\frac{\beta}{2N^2}\right)^4\right)$$

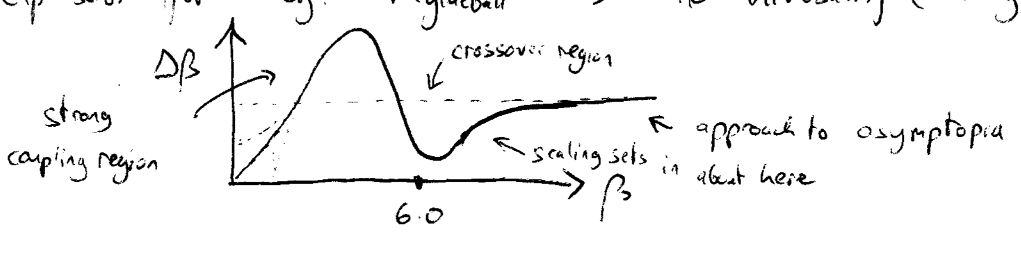
ie $$\Delta\beta(\beta) = \beta - \frac{\beta^4}{(2N)^3}$$

Should probably discard $O(\beta^4)$ to be consistent with initial approx.

$\boxed{\text{Points to discuss}}$

(i) we would have got a different answer had we started from a strong-coupling expression for eg. $M_{glueball}$ $\Rightarrow$ no universality @ strong coupling

(ii) full curve:



Solution to Exercise

(vii)

Fourier transforms (on ∞ lattice k is a continuous variable)

$$\psi(x) = \int_{-\pi/a}^{\pi/a} d^4k \; \psi(k)\, e^{+ik.x}$$

$$\bar{\psi}(x) = \int_{-\pi/a}^{\pi/a} d^4k \; \bar{\psi}(k)\, e^{-ik.x}$$

Free fermion action

$$\Rightarrow \quad S = a^4 \sum_{x\mu} \int_k \int_{k'} \frac{1}{2a}\, \bar{\psi}(k) \left\{ \left[\gamma_\mu \left(e^{+ik'_\mu a} - e^{-ik'_\mu a} \right) - r \left(e^{+ik'_\mu a} + e^{-ik'_\mu a} - 2 \right) \right] + m \right\} \psi(k')\, e^{i(k'-k).x}$$

Now, $\sum_x a^4 e^{-i(k-k').x} \quad \propto \quad \delta^4(k-k')$

$$\Rightarrow \quad S = \int_k \bar{\psi}(k) \left[\sum_\mu \frac{i}{a} \gamma_\mu \sin k_\mu a + \frac{r}{a} \left(1 - \cos k_\mu a \right) \right) \psi(k) + m \bar{\psi}(k) \psi(k)$$

ie $\quad S_F(k) = \left(\sum_\mu \frac{i}{a} \gamma_\mu \sin k_\mu a + \sum_\mu \frac{r}{a} \left(1 - \cos k_\mu a \right) + m \right)^{-1}$

↑ "momentum-dependent mass"

$\Rightarrow$ Long wavelength expansion @ $k = (0,0,0,0)$: $\quad S_F^{-1} = i\gamma_\mu k_\mu + m + O(ra)$

$\Rightarrow$ continuum form

But @ $k = (\frac{\pi}{a},0,0,0)$ $\quad S_F^{-1} = i\gamma_\mu k_\mu + \left(m + \frac{2r}{a} \right) + O(ra)$

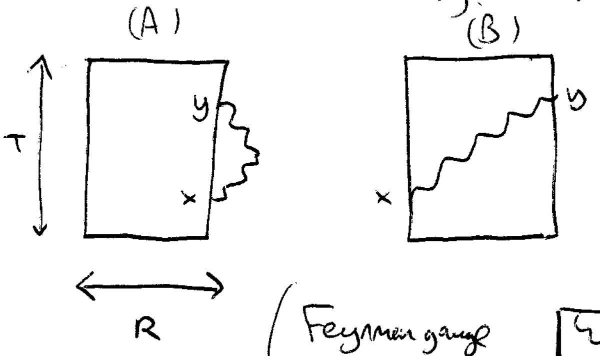
$\Rightarrow\Rightarrow$

1 species with mass	m	
4 " " "	$m + \frac{2r}{a}$	All become massive and decouple in $a \to 0$ limit
6 " " "	$m + \frac{4r}{a}$	
4 " " "	$m + \frac{6r}{a}$	
1 " " "	$m + \frac{8r}{a}$	

Wilson fermions violate the third N-N condition – ie the r term explicitly breaks the axial symmetry. It can be shown that if the axial Ward identity is recomputed, the correct anomaly emerges <u>independent</u> of numerical value of r. eg. Karsten & Smit, Nucl. Phys. <u>B183</u> (1981) 103

Solution to Exercise

(viii) We're actually evaluating Feynman diagrams in real space



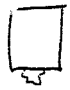
recall

$$v(x-y) = v(0)\,\delta_{xy} + v'(x-y)$$

$$v'(x-y) = \frac{1}{4\pi^2}\frac{1}{|x-y|^2} \quad \text{for } x \neq y$$

(Feynman gauge forbids )

$$\text{(A)} \quad -\frac{1}{2}e^2 \cdot \underbrace{2}_{\text{2 arms of loop}} \cdot \int_0^T dx \int_0^T dy\; v(x-y) \quad + \quad (T \leftrightarrow R)$$

ie 

$$= -\frac{1}{2}e^2 \cdot \frac{4}{4\pi^2}\left[\int_a^T dx \int_0^{x-a} dy \frac{1}{(x-y)^2}\right] - \frac{1}{2}e^2\, 2T\, v(0) + (T \leftrightarrow R)$$

$$= -\frac{e^2}{2\pi^2}\left[\frac{T}{a} - 1 - \ln\left(\frac{T}{a}\right)\right] - e^2 T\, v(0) + (T \leftrightarrow R)$$

x & y summed around all loop; extra − sign because $dx = -dy$ round Γ

$$\text{(B)} \quad = -\frac{1}{2}e^2 \cdot \frac{2}{4\pi^2}\int_0^T dx \int_0^T dy \frac{-1}{R^2 + (x-y)^2} \quad + \quad (T \leftrightarrow R)$$

$$= +\frac{e^2}{4\pi^2}\int_0^T dx\, \frac{1}{R}\left(\tan^{-1}\frac{x}{R} - \tan^{-1}\left(\frac{x-T}{R}\right)\right) \quad + \quad (T \leftrightarrow R)$$

Now use $\displaystyle\int \frac{1}{R}\tan^{-1}\frac{x}{R} = \frac{x}{R}\tan^{-1}\frac{x}{R} - \frac{1}{2}\ln(x^2 + R^2)$

$$\Rightarrow \quad = +\frac{e^2}{2\pi^2}\left[\frac{T}{R}\tan^{-1}\frac{T}{R} - \frac{1}{2}\ln\left(1 + \frac{T^2}{R^2}\right)\right] + (T \leftrightarrow R)$$

Add everything up & take limit $T \gg R$, (project onto lowest energy state in transfer matrix) then exponentiate

$$\Rightarrow \langle W(R,T)\rangle = \exp\left(\underbrace{-\frac{1}{2}e^2\left(v(0) + \frac{1}{2\pi^2 a}\right)(2(T+R))}_{\substack{\text{perimeter term} \\ \Leftrightarrow \text{ self energy of current}}} + \underbrace{\frac{e^2}{4\pi R}\cdot T}_{\text{Coulomb term}} + \underbrace{\frac{e^2}{2\pi^2}\left[\ln\left(\frac{RT}{a^2}\right) + 2\right]}_{\text{subsubleading}}\right)$$