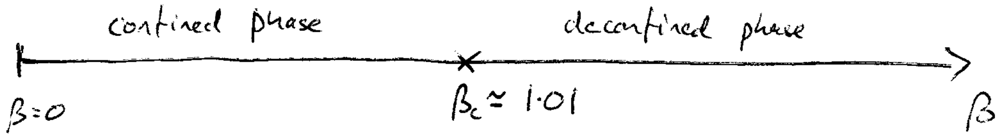
Electromagnetic Duality

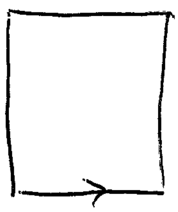
If you've thought about it, you would have realised by now that the strong coupling arguments I presented to demonstrate color confinement in $SU(N)$ gauge theory apply equally well in $U(1)$ LGT. But we know that QED is not a confining theory.

Solution: There is a phase separation in $U(1)$ LGT [Guth, Phys. Rev. D21 (1980) 2291]



Exercise: Calculate $\langle W(R,T)\rangle$ in the weak coupling phase for $U(1)$.
With $U_\mu(x) = \exp(ie A_\mu(x))$, we have

$$\langle W(R,T)\rangle \simeq \frac{1}{Z}\int DA_\mu \exp\left(-\frac{1}{4}\int F_{\mu\nu}^2 + ie\oint_\Gamma A_\mu . dx_\mu\right)$$



Γ

At weak coupling, can ignore all quantum fluctuations except the Gaussian ones ie. $\langle A_\mu(x) A_\nu(y)\rangle = \delta_{\mu\nu}\, v(x-y)$ (Feynman gauge)

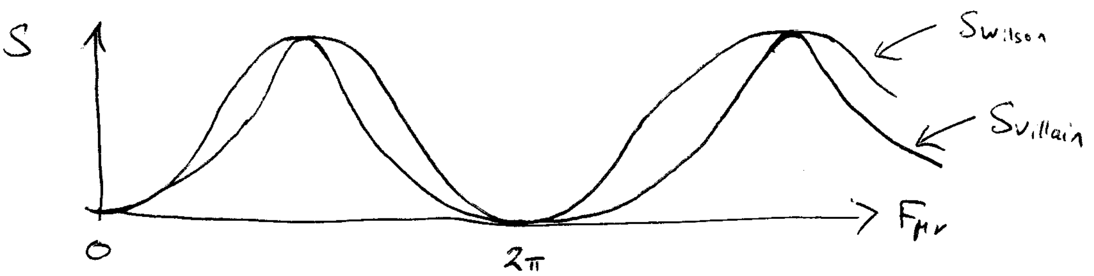
with $\begin{cases} v(x-y) = v(0)\,\delta_{xy} + v'(x-y) \\ v'(x-y) \simeq \dfrac{1}{4\pi^2}\dfrac{1}{|x-y|^2} \text{ if } |x-y|>a, \ 0 \text{ otherwise} \end{cases}$

Lattice Coulomb propagator $\Delta_\rho^+\Delta_\rho^- v(x) = -\delta(x)$

$$\Rightarrow \quad \langle W(R,T)\rangle \simeq \exp\left(-\frac{1}{2}e^2\oint_\Gamma\oint_\Gamma \langle A_\mu(x)A_\nu(y)\rangle . dx_\mu\, dy_\nu\right)$$

Evaluate $\langle W(R,T)\rangle$ for $T \gg R$ to show the leading decay $e^{-\# P}$, where $P = 2(T+R)$ is the perimeter of the loop, and the sub-leading decay $\exp\left(\dfrac{e^2}{4\pi R}T\right)$ ie. Coulomb's law.

Is there any way of understanding the phase transition?
$U(1)$ LGT is often called "periodic QED"
ie $\quad S_{\text{Wilson}} = \sum_{x\,\mu<\nu} 1 - \cos(F_{\mu\nu}(x))$ with $F_{\mu\nu}(x) = \Delta_\mu^+\theta_\nu(x) - \Delta_\nu^+\theta_\mu(x)$
ie sum of real link angles round plaquette.



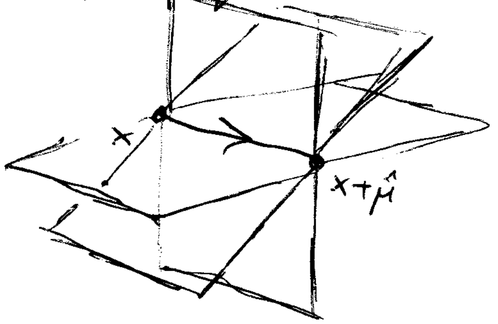
Often an alternative action, with the same periodicity & long-wavelength limit is used: the <u>Villain</u> action $\Rightarrow$ (Banks, Myerson & Kogut Nucl. Phys. <u>B129</u> (1977) 493)

$$Z_{Villain} = \int D\theta \sum_{\{n\}} \exp\left(\frac{i}{2} n_{\mu\nu}(x) F_{\mu\nu}(x) - \frac{e^2}{4} n^2_{\mu\nu}(x)\right)$$

where $n_{\mu\nu}(x)$ are **constrained** integer-valued variables defined on plaquettes. The Villain partition function can be understood (Sums on x, μ, ν understood) as the convolution **of** the gaussian weight $e^{-F_{\mu\nu}^2}$ with a periodic array of δ-functions$^{\text{PTO}}$. Since the Boltzmann weight is now linear in $\theta_\mu(x)$, $\int D\theta$ may be performed analytically:

$$Z_{Villain} = \sum_{\{n\}} \delta(\Delta^-_\nu n_{\nu\mu}) \exp\left(-\frac{e^2}{4} n_{\mu\nu}{}^2\right)$$

(The constraint arises because of the factor $e^{i\theta_\mu(x)\Delta^-_\nu n_{\nu\mu}} \Rightarrow$ generates a δ-fn on $\int D\theta$). Geometrically, it means the sum of plaquette variables bordering each link must vanish (3 sets in 4 dimensions!)



In general we don't like constraints in path integrals – however it is possible to resolve the constraint by a change of variables $n_{\mu\nu}(x) = \epsilon_{\mu\nu\lambda\kappa} \Delta^+_\lambda l_\kappa(\tilde{x}+\hat{\nu})$.

$$\Rightarrow \quad \Delta^-_\nu n_{\mu\nu}(x) = \epsilon_{\mu\nu\lambda\kappa} \Delta^-_\nu \Delta^+_\lambda l_\kappa(\tilde{x}+\hat{\nu}) = \epsilon_{\mu\nu\lambda\kappa} \Delta^+_\nu \Delta^+_\lambda l_\kappa(\tilde{x}) = 0$$

ie. n is **the** 4-dimensional **curl** of an integer-valued variable l

Here (i) $\Delta^+_\mu f = f(x+\hat{\mu}) - f(x)$; $\Delta^-_\mu f = f(x) - f(x-\hat{\mu})$

(ii) $\tilde{x}$ is a site on the <u>dual</u> lattice, ie an equivalent hypercubic with sites at $(x_1+\frac{1}{2}, x_2+\frac{1}{2}, x_3+\frac{1}{2}, x_4+\frac{1}{2})$ which interpenetrates the first.

$$f(z) = \int dx \; f(x) \; \delta(z-x)$$

$$f(z-m) = \int dx \; f(x) \; \delta(z-x-m)$$

$$\sum_m f(z-m) = \sum_m \int dx \; f(x) \, \delta(z-x-m)$$

$$f(x) = e^{-x^2}$$

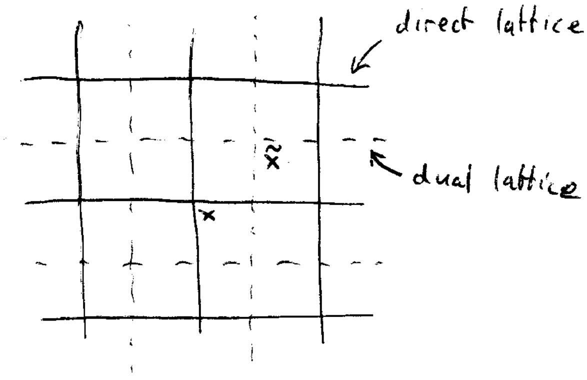
$$\Rightarrow \quad \sum_m e^{-(z-m)^2} = \sum_n \int dx \; e^{-x^2} \; e^{2\pi i(z-x)n}$$

$$= \sum_n e^{2\pi i z n} \int dx \; e^{-x^2 - 2\pi i x n}$$

$$= \sum_n e^{2\pi i z n} \int dx \; \exp\left[-(x+\pi i n)^2 - \pi^2 n^2\right]$$

$$\sum_m e^{-(z-m)^2} \;\; \alpha \;\; \sum_n e^{2\pi i z n - \pi^2 n^2}$$

eg. in 2 dimensions



In four dimensions there is a natural mapping

direct lattice		dual lattice
site	$\Longleftrightarrow$	hypercube
link	$\Longleftrightarrow$	cube
plaquette	$\Longleftrightarrow$	plaquette
cube	$\Longleftrightarrow$	link
hypercube	$\Longleftrightarrow$	site

$\Rightarrow$ Z_{Villain} can now be written in terms of dual variables:

$$Z_{\text{Villain}} = \sum_{\{\ell\}} \exp\left(-\frac{e^2}{4} \left(\Delta_\mu^+ \ell_\nu(\tilde{x}) - \Delta_\nu^+ \ell_\mu(\tilde{x}) \right)^2 \right)$$

$\Rightarrow$ a sort of "QED" with discrete-valued fields

Next step: reexpress Z using <u>Poisson</u> <u>resummation</u>, viz:

$$\sum_{\{\ell\}} f(\ell) = \sum_{\{m\}} \int_{-\infty}^{\infty} d\phi \, f(\phi) \, e^{2\pi i m \phi}$$

where m are integers, and ϕ real valued ~~fields~~ fields, both now defined on dual links.

$$Z_{\text{Villain}} = \sum_{\{m\}} \int D\phi \, \exp\left(-\frac{e^2}{4} \left(\Delta_\mu^+ \phi_\nu(\tilde{x}) - \Delta_\nu^+ \phi_\mu(\tilde{x}) \right)^2 + 2\pi i \, m_\mu(\tilde{x}) \phi_\mu(\tilde{x}) \right)$$

Now integral is Gaussian in ϕ, and we can evaluate it after a shift of integration variables (i.e. complete the square). $\Rightarrow$ Fixing Feynman gauge, we get

$$Z_{\text{Villain}} = \int D\phi' \exp\left(-\frac{e^2}{4} \left(\Delta_\mu^+ \phi_\nu'(\tilde{x}) - \Delta_\nu^+ \phi_\mu'(\tilde{x}) \right)^2 \right) \quad \text{free photons}$$

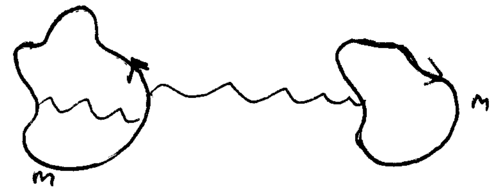
$$\times \sum_{\{m\}} \exp\left(\sum_{\tilde{x}\tilde{y}\mu} -\frac{g^2}{2} m_\mu(\tilde{x}) \, v(\tilde{x}-\tilde{y}) \, m_\mu(\tilde{y}) \right)$$

where v is the lattice Coulomb propagator. The m_μ obey a constraint $\Delta_\mu^- m_\mu(\tilde{x}) = 0$ (since get a term $e^{i\phi_\mu \Delta_\mu^- m_\mu}$ after a "gauge transformation" on ϕ)

$\Rightarrow$ hence m form integer-charged loops of conserved current

$\Rightarrow$ "Coulomb loop gas"

ie m form loop excitations interacting via Coulomb exchange.



The coupling $g = \frac{2\pi}{e}$, ie $\boxed{eg = 2\pi}$

Dirac quantisation condition

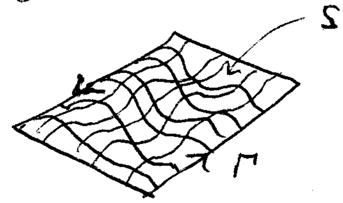
$\Rightarrow$ we interpret the m excitations as elements of magnetic current;
ie the loops are <u>monopole</u> — <u>anti-monopole</u> pairs

To display the magnetic property further, consider the interaction with a rectangle of electric current — ie. a Wilson loop $W(R,T)$
Using the same manipulations, we find for a current $j_\mu(x) = \begin{cases} 1 & \text{if link} \in \Gamma \\ 0 & \text{otherwise} \end{cases}$

$$Z[\Gamma] = Z_{photon} \cdot \exp\left(-\frac{e^2}{2}\, j_\mu(x)\, v(x-y)\, j_\mu(y)\right)$$
$$\times \sum_{\{m\}} \exp\left(-\frac{g^2}{2}\, m_\mu(\tilde{x})\, v(\tilde{x}-\tilde{y})\, m_\mu(\tilde{y})\right)$$
$$\times \exp\left(-2\pi i\, m_\mu(\tilde{x})\, v(\tilde{x}-\tilde{y})\, \frac{1}{2}\, \varepsilon_{\mu\lambda\rho\sigma}\, \Delta^+_\lambda S_{\rho\sigma}(y+\hat{\mu})\right)$$

ie a $j-j$ interaction of usual strength e^2 (Cf. today's exercise)
a $m-m$ interaction of dual strength g^2
a $m-j$ interaction of strength $2\pi = eg$

The plaquette variable $S_{\mu\nu}(x)$ is ± 1 if it belongs to a plaquette in a sheet spanning Γ, 0 otherwise



At first sight the dependence on S is alarming, since the spanning sheet is arbitrary, being only defined at its edge, ie. $\Delta^-_\nu S_{\nu\mu} = j_\mu$

However, it is straightforward to prove a <u>disentangling theorem</u>
ie. if $D_{\mu\nu}(\tilde{x}) \in \mathbb{Z}$, "Dirac sheet" variable spanning the m loop, then

$$m_\mu(\tilde{x})\, v(\tilde{x}-\tilde{y})\, \frac{1}{2}\, \varepsilon_{\mu\lambda\rho\sigma}\, \Delta_\lambda^+ S_{\rho\sigma}(y+\hat{\mu})$$

$$= -\, j_\sigma(y)\, v(y-x)\, \frac{1}{2}\, \varepsilon_{\sigma\lambda\tau\mu}\, \Delta_\lambda^+ D_{\tau\mu}(\tilde{x}-\hat{\lambda}-\hat{\mu}-\hat{\tau})$$

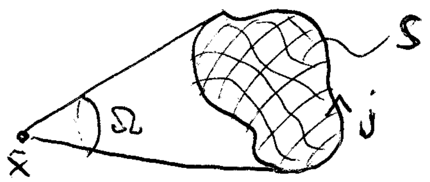
$$-\frac{1}{4}\, D_{\tau\mu}(\tilde{x})\, \varepsilon_{\sigma\lambda\tau\mu}\, S_{\rho\sigma}(x+\hat{\tau}+\hat{\mu})$$

ie. interaction is local in both m and j variables. The local contact interaction between D & S is an integer - the intersection number of the two surfaces in 4 dimensions. Hence $\exp(2\pi i\, \frac{1}{4} D_\varepsilon S) = 1$ so no dynamical effect. Our partition function thus describes interactions between conserved electric & magnetic currents.

(In 3d, (or the static limit of 4d) we could write the interaction $ig\, m_0(\tilde{x})\, V_{mag}(\tilde{x})$

with <u>magnetic scalar potential</u> $V_{mag}(\tilde{x}) = \sum_y -\frac{e}{2}\, \varepsilon_{ijk}\, \Delta_i^+ v_{3d}(\tilde{x}-\tilde{y})\, S_{jk}(y+\hat{o})$

$v_{3d}(r) \sim \frac{1}{4\pi r} \Rightarrow V_{mag} \simeq \frac{e}{4\pi}\int_S \frac{\underline{r}\cdot d\underline{S}}{r^3} = \frac{e\,\Omega}{4\pi}$ ← Solid angle subtended by S at $\tilde{x}$



Now geometrically Ω only defined modulo 4π
$\Rightarrow$ ambiguity of $ieg = 2\pi i$ in interaction
$\Rightarrow$ no dynamical effect.)

<u>Phase Transition</u>

From the exercise, we know that for a long loop of perimeter L, the excitation energy $E = \frac{g^2}{2}\, m\, v\, m \simeq \frac{1}{2} g^2 \left(v(0) + \frac{1}{2\pi^2} \right) L$

The entropy of such a loop: for a loop of length L there are approximately 7^L different configurations (ie. when tracing out such a non-backtracking loop we have a choice of $(2d-1)$ directions to hop at each node $\Rightarrow$ $S \simeq L \ln 7$

$\Rightarrow$ Expect a proliferation of loops when $F(L) = E(L) - S(L)$ changes sign:

ie. "Monopole condensation" at $\frac{1}{2} g_c^2 \left(v(0) + \frac{1}{2\pi^2} \right) \simeq \ln 7$

Now for 4-d lattice Coulomb propagator $v(0) \simeq 0.16$

$g^2 = 4\pi^2 \beta \Rightarrow \beta_c \simeq \frac{2 \ln 7}{4\pi^2 (0.16 + (2\pi^2)^{-1})} = 0.47$

Numerically, β_c for Villain model $\simeq 0.64$

ie for small e, g is large, hence m loops suppressed & are small & dilute

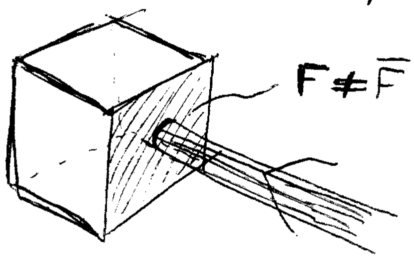
for large e, g is small, hence m loops dense & can grow without limit

For U(1) LGT (ie Wilson action) we can use an alternative definition of magnetic current (DeGrand Toussaint Phys. Rev. D22 (1980) 2478)

ie write $F_{\mu\nu}(x) = \bar{F}_{\mu\nu}(x) + 2\pi n_{\mu\nu}(x)$
with $\bar{F}_{\mu\nu} \in (-\pi, \pi]$, $n_{\mu\nu} \in \mathbb{Z}$

Then $m_\mu(\tilde{x}) = \frac{1}{2} \varepsilon_{\mu\nu\lambda\kappa} \Delta^+_\nu n_{\lambda\kappa}(x + \hat{\mu})$

ie.



$F \neq \bar{F} \Rightarrow$ Dirac string threading plaquette

Dirac strings cost no action in periodic QED.

Find spatial density of dual links with $m_\mu \neq 0$ increases dramatically across phase transition.

Monopoles & Confinement

Wilson loop $\langle W[\Gamma] \rangle \sim \exp\left(ie \oint_\Gamma A \cdot dx \right)$

$\sim \exp(ie\,\Phi(\Gamma))$ where Φ is magnetic flux penetrating the contour Γ.

Total flux from a monopole $= g = \frac{2\pi}{e} \Rightarrow$ presence of a monopole causes large fluctuations in value of $\langle W \rangle$ $\Rightarrow$

Area law decay follows if there ~~are~~ is a random gas or plasma of monopoles in vacuum, & if $M\bar{M}$ pairs are widely separated (ie gas of magnetic dipoles does not disorder $\langle W \rangle$ sufficiently)

$\Rightarrow$ need large extended monopole loop excitations

Numerically, one can factorise $U_\mu(x) = U_\mu(x)_{mon} \times U_\mu(x)_{photon}$

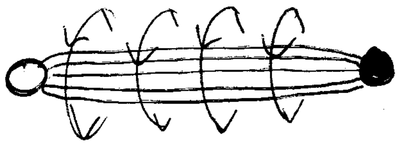
ie gauge configuration resulting from Coulomb gas of monopoles

superimposed "quantum" fluctuations

$\Rightarrow$ Find $U_\mu(x)_{mon}$ reproduces area law & string tension

$\Rightarrow$ "Dual Superconductor" picture of confinement

ie electric flux confined into tubes by



superconducting circulating magnetic current

Open Questions

(i) Is there an interesting non-trivial physical theory defined at the phase transition? One with light magnetic monopoles?

(ii) Can this mechanism be carried over to the non-abelian case ('t Hooft - Mandelstam)

$\Rightarrow$ this requires fixing a gauge & throwing away non-diagonal elements of the U_μ fields — <u>abelian projection</u>

Initial studies are promising.

And Finally ...

It's interesting that about 95% of the content of these lectures could have been presented 10 years ago. This partly reflects the fact that 6 lectures is barely enough to cover the basic ideas, and partly that in current LGT conceptual problems are being overtaken by technical ones.

Had I had more time I may have been able to cover one or more of the following ...

- Calculation of Standard Model Matrix Elements $\Rightarrow$ CKM matrix
- Calculation of $\alpha_s(M_Z)$ ie. precise numerical value for Λ_{QCD}
- Heavy Quark Physics, & Non-Relativistic QCD (NRQCD)
- QCD at non-zero baryon density
- Dynamical Symmetry Breaking in Strongly Coupled Theories (technicolor)
- Quantum Gravity / Random Surfaces
- Electroweak Baryogenesis
- Triviality Bounds on the Higgs Mass