Fermions

So far our discussion has centred on quarkless QCD. We can "bring it to life" by introducing Fermion fields $\psi(x)$, $\bar{\psi}(x)$ defined at the lattice sites. The lattice action:

$$S_f = a^4 \left[\sum_{x\mu} \frac{1}{2a} \left\{ \bar{\psi}(x) \gamma_\mu \psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu}) \gamma_\mu \psi(x) \right\} + m \sum_x \bar{\psi}(x) \psi(x) \right]$$

need symmetric difference for positive transfer matrix.

Here the ψ $\bar{\psi}$ are 4 component objects operated on by Euclidean Dirac matrices γ_μ, defined by (my convention chooses them hermitian):

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \; ; \quad \mathrm{tr}(\gamma_\mu \gamma_\nu) = 4\delta_{\mu\nu} \; ; \quad \mathrm{tr}\,\gamma_\mu = 0$$

Under $90°$ rotations $\psi(x) \mapsto \Lambda^{-1}_{\mu\nu} \psi(Rx)$, $\bar{\psi}(x) \mapsto \bar{\psi}(Rx) \Lambda_{\mu\nu}$ $\quad \Lambda_{\mu\nu} = \frac{1}{\sqrt{2}} \left(1 + \frac{1}{2}[\gamma_\mu, \gamma_\nu]\right)$

We can also introduce gauge transformations on the fermion fields: assume they transform in the fundamental representation: ie. as an N-vector of SU(N):

$$\psi(x) \mapsto \Omega(x) \psi(x) \; ; \quad \bar{\psi}(x) \mapsto \bar{\psi}(x) \Omega^+(x)$$

The gauge-invariant action is now

$$S_f = a^4 \left[\frac{1}{2a} \sum_{x\mu} \bar{\psi}(x) \gamma_\mu \left[U_\mu(x) \psi(x+\hat{\mu}) - U_\mu^+(x-\hat{\mu}) \psi(x-\hat{\mu}) \right) + m \sum_x \bar{\psi}(x) \psi(x) \right]$$

It is a trivial exercise to show that in the long wavelength limit

$$S \simeq \int d^4x \; \bar{\psi}(x) \left(\not{D} + m \right) \psi(x) + O(a^2)$$

with the <u>covariant derivative</u>: $\quad D_\mu \equiv \partial_\mu + igA_\mu$

and $\quad \not{D} = \not{\partial} + ig\not{A} \equiv \partial_\mu \gamma_\mu + ig A_\mu \gamma_\mu$

We have still to define the functional integration measure: to ensure Fermion statistics, ψ & $\bar{\psi}$ are defined as <u>Grassmann variables</u> ie. classically anti-commuting numbers

$$\{\psi, \psi\} = \{\bar{\psi}, \bar{\psi}\} = \{\psi, \bar{\psi}\} = 0$$

$$\Rightarrow \quad \psi_\alpha^2(x) = 0$$

with α a generic spin/color index

So how do we represent $\psi, \bar{\psi}$ on a computer? Answer: we don't!
We use the integration rules

$$\int d\psi_i \, 1 = 0 \qquad \int d\psi_i \, \psi_j = \delta_{ij}$$

to integrate the fermions out analytically. ie we define

$$Z = \int D\psi D\bar{\psi} DU \exp(-S_f[\psi,\bar{\psi},U]) \exp(-S_g[U])$$

$$\text{with } \int D\psi D\bar{\psi} = \int \prod_{x\alpha} d\psi_\alpha(x) \prod_{y\beta} d\bar{\psi}_\beta(y)$$

Note that S_f is bilinear in $\psi, \bar{\psi}$; ie $S_f = \bar{\psi}_i M_{ij} \psi_j$

where i,j range over space, spin & color indices, & M is thus a (big) matrix

Mathews-Salam formulae:

$$\int D\psi D\bar{\psi} \exp(-\bar{\psi}_i M_{ij} \psi_j) = \det M$$

$$\int D\psi D\bar{\psi} \, \psi_\ell \bar{\psi}_k \exp(-\bar{\psi}_i M_{ij} \psi_j) = (M^{-1})_{\ell k} \det M$$

(Obtain higher-point functions by Wick contraction...)

$$M_{xy} = \not{D}_{xy} + m\delta_{xy}$$

$$= \gamma_\mu \left[\delta_{y\,x+\hat{\mu}} U_\mu(x) - \delta_{y\,x-\hat{\mu}} U_\mu^\dagger(y) \right] + m\delta_{xy}$$

Note $\not{D}$ is anti-hermitian ie $\not{D}^\dagger = -\not{D}$
M is hermitian

$$\Rightarrow \quad Z = \int DU \det(\not{D}[U] + m) \exp(-S_g[U])$$

$$= \int DU \exp(-S_{eff}[U])$$

with $S_{eff}[U] = S_g[U] - \mathrm{tr} \ln (\not{D}[U] + m)$

ie. using $\ln \det A = \mathrm{tr} \ln A$

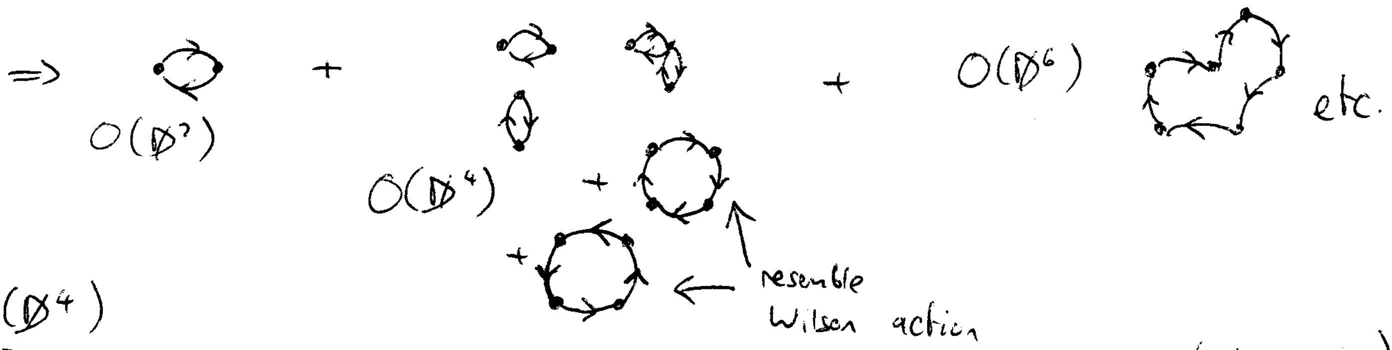
The effective action is highly non-local in the U fields eg. consider a $1/m$ or hopping parameter expansion

$$\mathrm{tr}\ln\left(\not{D}[U]+m\right) = \text{const.} + \mathrm{tr}\ln\left(1+\frac{\not{D}[U]}{m}\right)$$

$$= \text{const.} + \mathrm{tr}\left\{\frac{\not{D}[U]}{m} - \frac{\not{D}[U]^2}{2m^2} + \frac{\not{D}[U]^3}{3m^3} - \frac{\not{D}[U]^4}{4m^4} + \cdots\right]$$

on performing the trace, only contributions from closed paths of "hops"; ie only paths which come back to original site will lie on diagonal of $\not{D}^n$

$\Rightarrow$ all odd powers of $\not{D}$ vanish

$\Rightarrow$ 

at $O(\not{D}^4)$

For each plaquette there are 4 contributions (4 possible starting sites)

going either way $\Rightarrow$ numerical factor $4\times\frac{\not{D}^4}{4m^4} = -\frac{1}{(2m)^4}\left(U_\square + U_\square^\dagger\right)$

2+ signs, 2− signs from hopping $\Rightarrow$ $+$

$\mathrm{tr}\left(\gamma_\mu\gamma_\nu\gamma_\mu\gamma_\nu\right) = -4$

$\Rightarrow$ get a contribution to S_{eff} $-\frac{1}{4m^4}\,\mathrm{tr}\left(U_\square + U_\square^\dagger\right)$

$$\Rightarrow \quad S_{eff} = -\frac{\beta}{2N}\,\mathrm{tr}\left(U_\square + U_\square^\dagger\right) - \frac{1}{4m^4}\,\mathrm{tr}\left(U_\square + U_\square^\dagger\right)$$

$\Rightarrow$ renormalisation of $\beta \mapsto \beta_R = \beta + \frac{N}{2m^4} > \beta$

cf. vacuum polarisation $g \mapsto g_R < g$

Notice that the particle "paths" emerging from this expansion naturally generate Wilson loops $\Leftrightarrow$ justifies in retrospect our identification of $W(R,T)$ as (virtual) q-$\bar{q}$ pair

Simulation of Fermions

Direct evaluation of the fermion determinant would require $N! \sim e^N$ computations per update on a system of volume N — simply not on!
can be done in N^3 operations.

Fortunately one can use a trick — pseudofermions; ie generate determinant using auxiliary boson fields, with action

$$S_{pseud} \sim \Phi^+ (M^+M)^{-1} \Phi$$

$$\Rightarrow \int D\Phi D\Phi^+ \Rightarrow \det(M^+M) \Rightarrow 2 \text{ flavors of fermion}$$

$$\int D\Phi D\Phi^+ \exp(-\Phi^+ A \Phi) = (\det A)^{-1} \quad (\text{multi-dimensional Gaussian})$$

This method simply requires inversion of the matrix, or more precisely, one solution of $(M^+M)X = \Phi$ per update

$$\Rightarrow O(N^2) \text{ computations per update}$$

This is still a large amount of work — typically requiring 95% of the cpu. Many lattice theorists are thus tempted to make the dramatic, (and uncontrolled) quenched approximation

$$\text{ie set } \det M \equiv 1$$

$\Rightarrow$ treats "valence quarks" but ignores "sea quarks" ie. virtual fermion loops $\Rightarrow$ breaks unitarity of theory.

Hadron Spectroscopy

Following the glueball calculation, we "create" a generic meson from the vacuum using operator $\bar\psi(x)\Gamma\psi(x)$ where Γ is some Dirac matrix governing the quantum numbers of the meson. eg. $\Gamma = \gamma_5 \Rightarrow$ pion

$\gamma_\mu \Rightarrow$ rho

$\gamma_\mu\gamma_5 \Rightarrow a_1$ etc.

$\Rightarrow$ calculate meson propagator

$$G(x,y) = \langle \bar\psi(x)\Gamma\psi(x)\,\bar\psi(y)\Gamma\psi(y)\rangle$$

Wick contract $= \langle \Gamma\psi(x)\bar\psi(y)\,\Gamma\,\psi(y)\bar\psi(x)\rangle$

$$= \langle \mathrm{tr}\, \Gamma\, S_F(x,y)\, \Gamma\, S_F(y,x)\rangle$$

where $S_F(x,y)$ = fermion propagator $\langle\psi(x)\bar\psi(y)\rangle = M^{-1}_{xy}$

N.B. $\gamma_5 M^{-1}_{yx}\gamma_5 = (M^{-1}_{xy})^+$

$$\text{ie } G(x,y) = \langle \mathrm{tr}\, \Gamma M^{-1}_{xy}\, \Gamma M^{-1}_{yx}\rangle$$

$$\sum_{\underline{x}\,\underline{y}} G(\underline{x}, x_4 ; \underline{y}, y_4) \propto e^{-m|x_4-y_4|} \quad \text{as } |x_4-y_4| \nearrow \infty$$

$\Rightarrow$ extract meson mass

N.B. γ_5-hermiticity $\Rightarrow$ pion is lightest meson

A baryon may be "created" using a 3-quark operator,

eg. nucleon $N_\gamma(x) = \varepsilon_{ijk}\, \psi_{i\alpha} (C\gamma_5)_{\alpha\beta}\, \psi_{j\beta}\, \psi_{k\gamma}$

with i,j,k color indices, $\alpha\beta\gamma$ spin indices, and $C\gamma_\mu = -\gamma_\mu^* C$

The major problem left is to reach the <u>chiral limit</u> ie bare quark mass $m \to 0$. This is hard for 2 reasons:

(i) numerical: M becomes ill-conditioned as its diagonal elements $\propto m$

(ii) physical: PCAC hypothesis $\Rightarrow$ $m_\pi^2 \propto m$ so pion mass vanishes in chiral limit much faster than masses of other hadrons. Therefore pion correlation length ξ_π increases much faster; thus m must be kept large in order to avoid finite volume corrections

<u>Conceptual Problems</u>

Consider a change in integration variables in partition function:

$$\psi(x) \mapsto \psi'(x) = \psi(x) + i\alpha(x)\gamma_5\, \psi(x)$$
$$\bar\psi(x) \mapsto \bar\psi'(x) = \bar\psi(x) + i\alpha(x)\bar\psi(x)\gamma_5$$

Now, $D\psi' D\bar\psi' = D\psi D\bar\psi$ ie Jacobian $= 1$

(Proof: $J \propto \det(1+i\alpha\gamma_5) = \exp \operatorname{tr} \ln(1+i\alpha\gamma_5) = 1 + O(\alpha^2)$)

But $S_f[\psi,\bar\psi,U] \mapsto S_F[\psi',\bar\psi',U] + i\alpha(x)\left[\Delta_\mu^- J_{\mu 5}(x) - 2m\bar\psi(x)\gamma_5\psi(x)\right]$

with $\Delta_\mu^- f(x) \equiv f(x) - f(x-\hat\mu)$

$$J_{\mu 5}(x) = \frac{1}{2}\left[\bar\psi(x)\gamma_\mu\gamma_5 U_\mu(x)\psi(x+\hat\mu) + \bar\psi(x+\hat\mu)\gamma_\mu\gamma_5 U_\mu^\dagger(x)\psi(x)\right]$$

Since Z invariant under a change of variable:

$$\langle \Delta_\mu^- J_{\mu 5}(x) \rangle = 2m \langle \bar{\psi}(x)\gamma_5 \psi(x) \rangle$$

Axial Ward Identity

But in continuum QFT:

$$\partial_\mu J_{\mu 5}(x) = 2m\, \bar{\psi}(x)\gamma_5\psi(x) + \frac{g^2}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x)$$

ie. an extra term – the $U(1)$ axial anomaly
(Adler, Bell & Jackiw)

Why is **there** **no** anomaly on the lattice? Consider the free fermion propagator in momentum space:

$$S_F(k) = \langle \psi(k)\bar{\psi}(k) \rangle = \left(\sum_\mu \frac{i\gamma_\mu}{a} \sin k_\mu a + m \right)^{-1} \qquad k \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right)$$

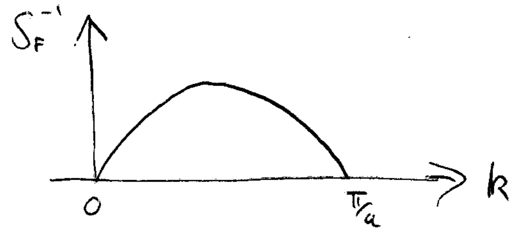
ie 1st Brillouin Zone

In the long wavelength limit $ka \to 0$ can expand @ $k=0$

$$S_F(k) = (ik_\mu\gamma_\mu + m + O(a^2))^{-1}$$

usual continuum form.

But:



$\sin k_\mu a$ vanishes at edge of Brillouin zone as well!

$\Rightarrow$ extra pole in propagator $\Rightarrow$ extra particle species!

eg. for the pole at $\bar{p} = (\pi/a, 0, 0, 0)$
get particle propagator $\tilde{S}_F(k) = (i(k-\bar{p})_\mu\gamma_\mu + m)^{-1}$

In d dimensions, find 2^d fermion species – the fermion doubling problem

So no anomaly, because 8 fermions have **+ve** axial charge
8 " " **−ve** " " } "parity doubling"

$\Rightarrow$ ABJ triangle diagram cancels

but eg. vacuum polarisation  gets multiplied by 16!

Nielsen - Ninomiya "No-Go" Theorem (Nucl. Phys. B185 (1981) 20; B193 (1981) 173; Phys Lett. 105B (1981) 219)

For fermion fields formulated on a regular lattice, parity doubling of fermion species is inevitable if the action is

(i) reflection positive (ie. Hamiltonian is hermitian)

(ii) local; ie interaction between $\bar{\psi}(x)$ and $\psi(y)$ decreases faster than $\frac{1}{|x-y|}$

(iii) has a global axial symmetry yielding fermions with discrete conserved axial charges

Exercise: The naive fermion action has the degeneracy between species **lifted** if a "Wilson term" (ITMA!) - Most commonly used method in lattice QCD.

ie
$$S_f \mapsto S_f - a^4 \cdot \frac{r}{2a} \sum_{x\mu} \left\{ \bar{\psi}(x)\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\psi(x) - 2\bar{\psi}(x)\psi(x) \right\}$$

Calculate the propagator in momentum space & display the degeneracy breaking. Which of the N-N conditions does the "Wilson fermion" violate?

Implications of the NN theorem:

There are no neutrinos (Weyl fermions) on the lattice

$\Rightarrow$ There is no lattice formulation of the Standard Model

$\Rightarrow$ There is no non-perturbative definition of the Standard Model

Is this a problem? I suggest yes: one current SM process requiring non-perturbative treatment is electroweak baryogenesis

ie. axial anomaly $\Leftrightarrow$ non-conservation of B-L

$\Rightarrow$ particle production by topologically non-trivial gauge field backgrounds at high T.

Ginsparg-Wilson Fermions (Gattringer & Lang ch. 7)

Currently the optimal solution for lattice fermions with a chiral symmetry, motivated by Renormalization Group blocking. — a new way to think about chiral symmetry.

Schematically: fermion action $= \bar\psi D \psi + m\bar\psi\psi$

Chiral symmetry $\Longleftrightarrow$ $\gamma_5 D + D\gamma_5 = 0$

GW proposal: modify this to

$$D\gamma_5 + \gamma_5 D = a D\gamma_5 D \quad (*)$$

RHS formally $O(a)$

$$\Rightarrow \gamma_5 D^{-1}_{xy} + D^{-1}_{xy}\gamma_5 = a\gamma_5\delta_{xy}$$

non-zero RHS manifests itself as a contact term in propagator $\Rightarrow$ irrelevant in long-wavelength limit?

GW fermions have a modified chiral symmetry:

$$\psi \mapsto \exp\left(i\alpha\gamma_5\left(1-\tfrac{a}{2}D\right)\right)\psi;\quad \bar\psi \mapsto \bar\psi\exp\left(i\alpha\left(1-\tfrac{aD}{2}\right)\gamma_5\right)$$

$$\Rightarrow \bar\psi D\psi \mapsto \bar\psi e^{i\alpha(1-\frac{aD}{2})\gamma_5} D e^{i\alpha\gamma_5(1-\frac{aD}{2})}\psi$$

$$= \bar\psi e^{i\alpha(1-\frac{aD}{2})\gamma_5} e^{-i\alpha(1-\frac{aD}{2})\gamma_5} D\psi = \bar\psi D\psi$$

using GW $(*)$

Recover desired continuum form in limit $aD\to 0$

ie $|D_{xy}| \sim e^{-\lambda|x-y|}$ with λ fixed in lattice units

ie $\lim_{a\to 0}\frac{\lambda}{a} > 0$

How is GW relation realised in practice?

(i) Overlap fermions $\quad D_{ov} = \frac{1}{a}\left[1 + \gamma_5 \, \mathrm{sgn}[H]\right]$

$H \equiv \gamma_5 A \quad$ A is the "kernel" operator $\sim \not{D}$

$$\gamma_5 A \gamma_5 = A^+ \Rightarrow H = H^+$$

$$\mathrm{sgn}[H] = U^+ \begin{pmatrix} \mathrm{sgn}(\lambda_1) & & \\ & \mathrm{sgn}(\lambda_2) & \\ & & \ddots \end{pmatrix} U = U^+ \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \ddots \end{pmatrix} U$$

$\Rightarrow D_{ov} = \frac{1}{a}\left(\mathbb{1} + \frac{\gamma_5 H}{\sqrt{H^2}}\right)$ locality of D_{ov} is <u>not</u> manifest

D_{ov} must be estimated numerically using polynomial/rational approx.

GW? $a D_{ov}\gamma_5 D_{ov}\gamma_5 = \frac{1}{a}\left(1+\gamma_5 \mathrm{sgn}(H)\right)\left(1+\mathrm{sgn}(H)\gamma_5\right)$

$$= \frac{1}{a}\left(1+\gamma_5 \mathrm{sgn}(H) + \mathrm{sgn}(H)\gamma_5 + 1\right) = D_{ov} + D_{ov}^+$$

$$\Rightarrow a D_{ov}\gamma_5 D_{ov} = D_{ov}\gamma_5 + \gamma_5 D_{ov}$$

Simplest choice: $A = D_W - M$ $\quad$ D_W is Wilson fermion operator

weak coupling $ap \to 0$: $\quad D_W \sim i\gamma_\mu p_\mu \quad$ near $p = (0,0,0,0)$

$$\mathrm{sgn}[H] = \frac{H}{(H^2)^{1/2}} \simeq \gamma_5 \frac{(i\not{p} - M)}{M}$$

$$\mathrm{sgn}[H] \simeq \gamma_5 \frac{(i\not{\tilde{p}} + 2 - M)}{2-M} \quad \text{near } p = (\pi, 0, 0, 0)$$

$$\tilde{p} = p - \pi$$

Massive overlap: $D_{ov}^m = \frac{(1+m)}{2} + \frac{(1-m)}{2}\gamma_5 \, \mathrm{sgn}[H]$

near $p=0 \Rightarrow D_{ov}^m \simeq i\not{p}\underbrace{\frac{(1-m)}{M}} + m$

$O(1)$ wavefn. renormalisation $\quad$ pole mass $\propto m$

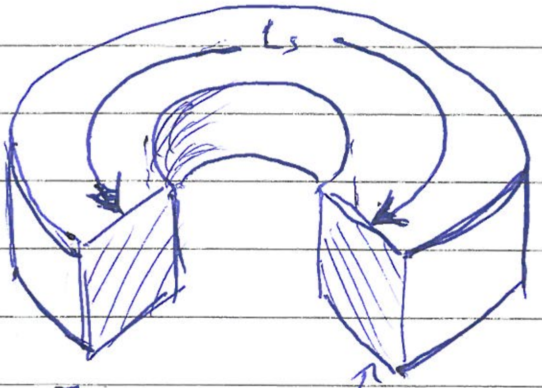
near $p = \pi \Rightarrow D_{ov}^m \simeq \frac{i(1-m)\not{\tilde{p}}}{2(2-M)} + 1 \leftarrow O(a^{-1})$ mass decouples

(ii) Domain Wall Fermions

$$S_{DW} = \sum_{x,s} \delta_{ss'} \bar{\Psi}_{xs} (D_W - M)_{xy} \Psi_{ys'} + \delta_{xy} \bar{\Psi}_{xs} D_{5\,ss'} \Psi_{ys'}$$

$$(D_5)_{ss'} = -\left[P_- \delta_{s+1,s'} (1-\delta_{s' L_s}) + P_+ \delta_{s-1,s'} (1-\delta_{s',1}) \right] + \delta_{s's}$$

$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ chiral projectors



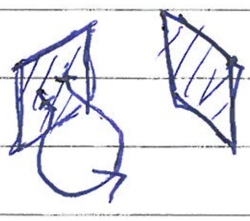
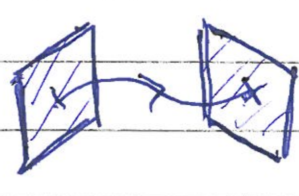
Can show (Kaplan) approximate zero modes of D_{DW} localised on "domainwalls" at $s = 1, L_s$ with $\gamma_5 |\Psi\rangle = \pm |\Psi\rangle$ as $L_s \to \infty$

open boundary conditions in s-direction $\Rightarrow$ "Physical" fields in 4d target space

$$\psi(x) = P_- \Psi(x,1) + P_+ \Psi(x, L_s)$$
$$\bar{\psi}(x) = \bar{\Psi}(x, L_s) P_- + \bar{\Psi}(x,1) P_+$$

Gauge fields $U_\mu(x,s)$ are taken to be constant throughout 4+1d bulk: $\partial_s U_\mu(x,s) = 0$

All physical Green functions formed from $\langle \Psi \;\bar{\Psi} \rangle$:

ie  or 

DWF mass term:

$$m \left[\bar{\Psi}(x, L_s) P_- \Psi(x,1) + \bar{\Psi}(x,1) P_+ \Psi(x, L_s) \right]$$

couples walls together.

Relation between Overlap & DWF:
(AD Kennedy hep-lat/0607038)

Can prove that $\det[D_{DW}^{-1}(1)\ D_{DW}(m)] \equiv \det D_{L_s}[H]$

describes regulator or "Pauli Villars" fermions cancelling bulk contribution

Truncated overlap operator

$$D_{L_s}[H] = \frac{1}{2}\left[(1+m) + (1-m)\gamma_5 \tanh(L_s \tanh^{-1}(H))\right]$$

with $H = \gamma_5 \dfrac{(D_w - M)}{[D_w + 2 - M]} \equiv \gamma_5 A$

A is Shamir kernel   has correct long wavelength limit for $M,\ 2-M \sim O(1)$

$$\lim_{L_s\to\infty} D_{L_s}[H] = \frac{1}{2}\left[(1+m) + (1-m)\gamma_5 \,\mathrm{sgn}[H]\right] = D_{ov}^M$$

obeys: $\gamma_5 D_{ov} + D_{ov}\gamma_5 = 2 D_{ov}\gamma_5 D_{ov}$

variant of (*)