# Transfer Matrix (Smit 2.1)

Important in linking Euclidean path integral with canonical QFT.
We will illustrate its derivation in quantum mechanics.

Consider a classical system described by

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \qquad \text{or} \qquad H = \frac{p^2}{2m} + V(x)$$

$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = 0$

where $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\qquad p = m\dot{x}, \quad \dot{p} = -\frac{dV}{dx}$

Passage to quantum theory: $p, x \mapsto \hat{p}, \hat{x}$ with $[\hat{x}, \hat{p}] = i\hbar$

Time evolution of wavefunctions given by unitary evolution operator

$$\hat{U}(t_1, t_2) = \exp[i \hat{H}(t_1 - t_2)/\hbar]$$

eg. $\psi(x, t_1) = \hat{U}(t_1, t_2)\, \psi(x, t_2)$ — (*) $\quad \hat{H} = \hat{H}(\hat{p}, \hat{x})$

Can also work with time-dependent c-numbers $x(t)$ using path integral.
Define a basis $|x\rangle$:

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x'|x\rangle = \delta(x' - x), \quad \int dx\, |x\rangle\langle x| = 1$$

plus $\langle x|p\rangle = e^{ipx}$ $\quad \int \frac{dp}{(2\pi)} |p\rangle\langle p| = 1 \quad \Rightarrow$ defines $|p\rangle$ states

Then in real time (ie. Minkowski metric) path integral is

$$\langle x_1 | \hat{U}(t_1, t_2) | x_2 \rangle = \int Dx \exp\left(i \frac{S(x)}{\hbar}\right) \qquad \text{— (**)}$$

where $S = \int_{t_2}^{t_1} L(x(t), \dot{x}(t))\,dt$, $\int Dx$ is integral over all $x(t)$ such that $x(t_1) = x_1$, $x(t_2) = x_2$

We want formal relation between (*) & (**)

Classical path $\delta S(x) = 0 \Rightarrow$ Euler-Lagrange eqn. $\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = 0$

if $\delta S/\hbar \sim O(1)$ need more robust definition, $\Rightarrow$ need to discretise time

eg. $\int Dx = \prod_{t_2 < t_i < t_1} \int dx(t_i)$, $\quad S = a \sum_{0 < n < \frac{t_1 - t_2}{a}} \frac{m}{2a^2}(x_{n+1} - x_n)^2 - \frac{1}{2}V(x_{n+1}) - \frac{1}{2}V(x_n)$

ie $\quad t - t_2 = na \quad , \quad 0 < n < \frac{t_1 - t_2}{a} - 1 = N - 1$

Now define a Transfer Operator $\hat{T}$:
generates time-evolution over one discrete jump of length $a$.

$$\langle x_1 | \hat{T} | x_2 \rangle = C \exp\left\{ ia \left[ \frac{M}{2a^2} (x_1 - x_2)^2 - \frac{1}{2} V(x_1) - \frac{1}{2} V(x_2) \right] \right\}$$

C-number matrix elements define Transfer Matrix
$\Rightarrow$ discretised path integral reads

$$\langle x' | \hat{U}(t', t'') | x'' \rangle = \int dx_1 \dots dx_{N-1}$$

$$\langle x' | \hat{T} | x_{N-1} \rangle \langle x_{N-1} | \hat{T}' | x_{N-2} \rangle \dots \langle x_1 | \hat{T} | x'' \rangle$$

$$\Rightarrow \hat{U} = \hat{T}^{N-1}$$

$$= C \int \left( \prod C dx \right) \exp\left[ \frac{iM}{2a} (x' - x_{N-1})^2 - \frac{ia}{2} V(x') - ia V(x_{N-1}) + \frac{iM}{2a} (x_{N-1} - x_{N-2})^2 \right.$$

$$\left. - ia V(x_{N-2}) + \dots + \frac{iM}{2a} (x_1 - x'')^2 - \frac{ia}{2} V(x'') \right] \equiv \int Dx \, e^{iS}$$

Equivalent to continuum form in $N \to \infty$ limit, when $x_n \to x(t)$
This integral provides a precise definition of "sum over paths" $\int Dx$.
Note typical path is very rough.

What is the expression for $\hat{T}$ in terms of $\hat{p}$ & $\hat{x}$?

Ansatz: $\hat{T} = \exp\left( -ia \frac{V(\hat{x})}{2} \right) \exp\left( -ia \frac{\hat{p}^2}{2m} \right) \exp\left( -ia \frac{V(\hat{x})}{2} \right)$

This reproduces above matrix elements if

$$\langle x_1 | \exp\left( -ia \frac{\hat{p}^2}{2m} \right) | x_2 \rangle = C e^{iM(x_1 - x_2)^2/2a}$$

$$= \int \frac{dp}{2\pi} \langle x_1 | \exp\left( -ia \frac{\hat{p}^2}{2m} \right) | p \rangle \langle p | x_2 \rangle = \int \frac{dp}{2\pi} e^{-iap^2/2M} e^{i(x_1 - x_2)p}$$

gaussian $\int$.

$$= \int \frac{dp}{2\pi} \exp -i \left( \sqrt{\frac{a}{2M}}\, p - \sqrt{\frac{M}{2a}} (x_1 - x_2) \right)^2 \exp iM \frac{(x_1 - x_2)^2}{2a} = \int \sqrt{\frac{M}{2\pi^2 ia}} \, du \, e^{-u^2} e^{iM\frac{(x_1 - x_2)^2}{2a}}$$

$\Rightarrow$ get correct transfer matrix with $C = \sqrt{\dfrac{M}{2\pi ia}}$

Since $\hat{T}$ composed of 3 unitary operators, can write $\hat{T} = \exp(-ia \hat{H})$
defines ~~unitary~~ Hermitian Hamiltonian $\hat{H}$ $\Rightarrow \hat{U} = \exp(-ia(N-1)\hat{H})$

for low energy states $\hat{T} \simeq 1 - ia\hat{H} \Rightarrow \hat{H} = \dfrac{\hat{p}^2}{2M} + V(\hat{x}) + O(a^2)$

Now continue analytically to imaginary time $t \to -it$

$a \to |a| e^{-i\pi/2}$

$$\Rightarrow \langle x' | \hat{U}(t_1, t_2) | x'' \rangle = |C| \int \left( \prod_n |C| dx_n \right) e^{-S}$$

$$S = +|a| \sum_{n=0}^{N-1} \left[ \frac{m}{2|a|^2} (x_{n+1} - x_n)^2 + \frac{1}{2} V(x_{n+1}) + \frac{1}{2} V(x_n) \right]$$

integrand has much better convergence properties.
cf. Euclidean path integral of QFT.

But we still have a transfer matrix!

$$\hat{T} = \exp\left( -a \frac{V(\hat{x})}{2} \right) \exp\left( -a \frac{\hat{p}^2}{2m} \right) \exp\left( -a \frac{V(\hat{x})}{2} \right) = e^{-a\hat{H}}$$

The eigenvalues of $\hat{T}$ and $\hat{H}$ are all positive in this formalism.

$$\hat{T} | i \rangle = e^{-aE_i} | i \rangle$$

Now consider a Euclidean correlation function for $x' \to \infty$, $x'' \to -\infty$

$$\langle A(t_1) A^+(t_2) \rangle = \sum_n \langle 0 | A | i \rangle e^{-NaE_i} \langle i | A^+ | 0 \rangle$$

$$= \sum_n \alpha_i e^{-NaE_i} \qquad \alpha_i = | \langle 0 | A | i \rangle |^2 \qquad |i\rangle \text{ t-independent}$$

for $N \to \infty$ this sum is dominated by state $|1\rangle$ with smallest $E_1$ such that $\langle 0 | A | 1 \rangle \neq 0$.

$$\Rightarrow \lim_{N \to \infty} \langle A(t_1) A^+(t_2) \rangle \propto e^{-E(t_1 - t_2)}$$

since $E$ is an eigenvalue of $\hat{H}$, can identify as an energy of a state. In this way we can calculate SPECTRAL QUANTITIES, ie masses or excitation energies, without a detailed knowledge of the $\alpha_i$.

Usually it's enough to know $\hat{T}$ exists, without specifying it in detail

Non-Zero Temperature: $Z = \mathrm{Tr}\, e^{-\hat{H}(t_+ - t_-)} = \int dx \langle x | e^{-H(t_+ - t_-)} | x \rangle = \mathrm{Tr}\, \hat{T}^N$

with $t_+ - t_- = Na$. But $Z$ is canonical partition function of statistical mechanics with temperature $T = (t_+ - t_-)^{-1}$ $(k_B = 1)$

$$\Rightarrow Z = \int_{pbc} Dx\, e^{-S} \quad \text{ie with } x(t_+) \equiv x(t_-)$$