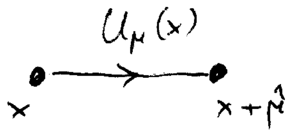
# Lattice Gauge Theory

Initially we will define LGT as we did for the Ising Model. We start with a 4-dimensional hypercubic lattice — each site $x$ can be labelled by 4 integers. On each link of the lattice, connecting sites $x$, $x+\hat{\mu}$ we define the dynamical variables $U_\mu(x)$

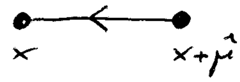



$U_\mu(x)$ is an element of a compact group $\mathcal{G}$, the gauge group

eg. $\mathcal{G} = Z_2$ ; $U_\mu(x) = \pm 1$ $\qquad$ $\mathcal{G} = U(1)$ ; $U_\mu(x) = e^{i\theta_\mu(x)}$

$\mathcal{G} = SU(N)$ ; $U_\mu(x)$ is a $N \times N$ matrix, with $U^{-1} = U^+$, $\det U = 1$

$U_\mu(x)$ is an oriented variable, ie



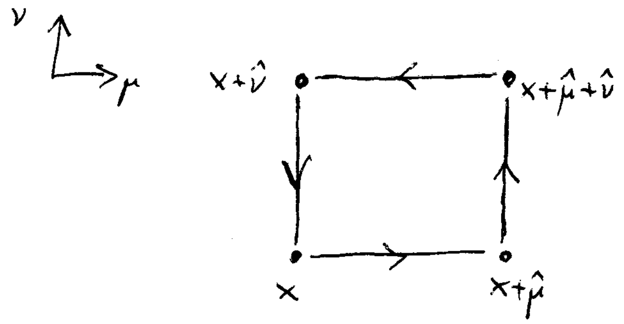

$= U_\mu^{-1}(x)$

Let's specialise to $\mathcal{G} = SU(N)$, and define the action

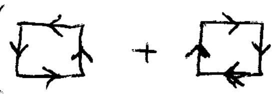
Wilson action (1974) $\quad S = \sum_{x\mu<\nu} -\frac{\beta}{N} \, \mathrm{Re}\,\mathrm{tr}\, U_{\mu\nu}(x)$

coupling constant ~ "inverse temperature"

where the plaquette variable $U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^+(x+\hat{\nu})\, U_\nu^+(x)$




Note $S = -\frac{\beta}{N}\, \mathrm{tr}\, \frac{U_{\mu\nu}(x) + U_{\mu\nu}^+(x)}{2}$

$\simeq \frac{1}{2}\Big($  $\Big)$

Why this form? : Consider the action of a local gauge transformation $\Omega(x) \in \mathcal{G}$, defined independently at each site

$$U_\mu(x) \mapsto \Omega(x)\, U_\mu(x)\, \Omega^+(x+\hat{\mu})$$

Under a local gauge transformation, $S$, or indeed any combination of link variables forming a closed path, is **invariant**:

$$U_{\mu\nu}(x) \mapsto \mathrm{tr}\left\{ \Omega(x)\, U_\mu(x)\, \Omega^\dagger(x+\hat\mu)\, \Omega(x+\hat\mu)\, U_\nu(x+\hat\nu)\, \Omega^\dagger(x+\hat\mu+\hat\nu) \right.$$
$$\left. \Omega(x+\hat\mu+\hat\nu)\, U_\mu^\dagger(x+\hat\nu)\, \Omega^\dagger(x+\hat\nu)\, \Omega(x+\hat\nu)\, U_\nu^\dagger(x)\, \Omega^\dagger(x) \right\}$$
$$= \mathrm{tr}\left\{ \Omega(x)\, U_{\mu\nu}(x)\, \Omega^\dagger(x) \right\} = \mathrm{tr}\, U_{\mu\nu}(x)$$

<u>Classical Continuum Limit</u>

Since $U_\mu(x) \in \mathcal{G}$, it is possible to express it as follows:

$$U_\mu(x) = \exp\left( i \frac{ga}{2} \lambda^a A_\mu^a(x) \right)$$

where: $a$ is the lattice spacing (measured in fm!)
$g$ will turn out to be a coupling constant

$A_\mu^a(x)$ is a real dimension 1 field defined on the link
— it will turn out to be the gauge potential field

$\lambda^a$ are traceless Hermitian generators of Lie Algebra of $\mathcal{G}$
& have matrix representations: eg $\lambda^a$ are $N\times N$ matrices for $U \in$ fundamental rep.

ie. $a = 1, \ldots, N^2-1$ ; $[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c$
$\mathrm{tr}(\lambda^a \lambda^b) = 2\,\delta^{ab}$

N.B. factors of 2 are convention-dependent - beware!

$$\Rightarrow U_{\mu\nu}(x) = \exp\left(\frac{iga}{2}\lambda^a A_\mu^a(x)\right) \exp\left(\frac{iga}{2}\lambda^b A_\nu^b(x+\hat\mu)\right)$$
$$\times \exp\left(-\frac{iga}{2}\lambda^c A_\mu^c(x+\hat\nu)\right) \exp\left(-\frac{iga}{2}\lambda^d A_\nu^d(x)\right)$$

Now let us Taylor expand: ie. $A_\mu(x+\hat\nu) = A_\mu(x) + a\,\partial_\nu A_\mu(x) + O(\partial A)^2$

Meaning what? Taylor expansions must be in a <u>dimensionless</u> parameter
ie **$a\partial$** in this case. $\partial(aA) \ll$ **$A$**
ie. $A$ is slowly varying **over scale** of a lattice spacing

$$\Rightarrow \quad U_{\mu\nu}(x) =$$

$$\exp\left(\frac{iga}{2}\lambda^a A_\mu^a(x)\right)\exp\left(\frac{iga}{2}\left\{\lambda^b A_\nu^b(x) + a\lambda^b \partial_\mu A_\nu^b(x)\right\}\right)$$

$$\times \exp\left(-\frac{iga}{2}\left\{\lambda^c A_\mu^c(x) + a\lambda^c \partial_\nu A_\mu^c(x)\right\}\right)\exp\left(-\frac{iga}{2}\lambda^d A_\nu^d(x)\right)$$

To combine the exponentials, recalling that they are matrices, we use the Baker-Campbell-Hausdorff formula:

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + O(A^3)\right)$$

$\Rightarrow\Rightarrow$

$$U_{\mu\nu}(x) = \exp\left[\frac{iga^2}{2}\lambda^a\left\{\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right\} - \frac{g^2a^2}{4}A_\mu^a A_\nu^b[\lambda^a,\lambda^b] + O(a^3A^3, a^3\partial A^2 + a^3\partial^2 A)\right]$$

$$= \exp\left(\frac{iga^2}{2}\lambda^a F_{\mu\nu}^a(x) + O(a^3)\right)$$

with $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$

ie. with $F_{\mu\nu}^a$ the continuum Yang-Mills field strength tensor

Now expand exponential:

$$U_{\mu\nu}(x) = 1 + \frac{iga^2}{2}\lambda^a F_{\mu\nu}^a(x) - \frac{g^2a^4}{8}\lambda^a\lambda^b F_{\mu\nu}^a F_{\mu\nu}^b + O(a^5)$$

(No sum on $\mu\nu$)

double counting on implied sum on $\mu\nu$

? $\mathrm{tr}\,\lambda^a\lambda^b = 2\delta^{ab}$

hence $S = \sum_{\mu\nu x} -\frac{\beta}{N}\,\mathrm{Re}\,\mathrm{tr}\,U_{\mu\nu}(x)$

$$= \sum_x\left\{-\beta + 0 + \beta\frac{a^4g^2}{8N}F_{\mu\nu}^a F_{\mu\nu}^a + O(a^5)\right\}$$

irrelevant constant $\Rightarrow F \to F + \delta F$

$\lambda^a$ traceless

Now, in small $a$ limit $\sum_x a^4 \sim \int d^4x$

$$\Rightarrow \quad S = \text{constant} + \int d^4x\,\frac{1}{4}F_{\mu\nu}^a(x)F_{\mu\nu}^a(x) + O(a)$$

with the identification $\beta = \frac{2N}{g^2}$

ie. $S \simeq$ continuum Yang-Mills action

With more work, it can be shown that the correction is in fact $O(a^2)$ (Just be glad it isn't an exercise ... )

Notes :

- The higher order terms are suppressed by $O(a\partial)$ and $O(aA)$ ie our Taylor expansion is justified if $aD_\mu A$ is small

- Continuum **Yang-Mills** theory is defined on the Lie Algebra of $\mathcal{G}$ and is unique. LGT is defined in terms of $\mathcal{G}$ itself, & depends on global properties of group, which representations are present, etc.

  eg. $\mathcal{G} = SU(2)$ or $SO(3)$

  Identical Lie algebras $\Rightarrow$ identical continuum theories

  however, $SO(3)$ contains integer spin reps only

  $SU(2)$ . " $\frac{1}{2}$ " " " as well

  eg. $-\mathbb{1} \in SU(2)$, but not $SO(3)$ $\Rightarrow$ distinct LGT's

  $SO(3) \simeq SU(2)/Z_2$

- Whilst it may be possible to treat $aA$ as a small parameter locally, it is impossible to enforce this condition (through choice of gauge) over arbitrary separations on the lattice ie. there is a scale $\xi \sim \frac{1}{\Lambda_{QCD}}$ beyond which it is impossible to rotate all $U_\mu$ to near the identity $\mapsto$ a formalism based on $A$, ie perturbation theory, is bound to fail

To complete our definition of LGT, we need a measure to define the generating function $Z$ in terms of $S$.

ie $$Z = \int DU \exp(-S[U])$$

In fact, $\int DU$ can be written $\int \prod_{\text{links}} dU$

where $dU$ is the invariant or <u>Haar</u> measure over the group manifold

It is defined by its properties:

left & right invariance (no favoured point on group manifold)

$$\int dU\, f(U) = \int dU\, f(\Omega U) = \int dU\, f(U\Omega)$$

where $\Omega$ is an arbitrary constant element of $\mathcal{G}$

normalisation $$\int dU\, 1 = 1$$

If $\mathcal{G}$ is compact, then any element can be specified by $n$ real parameters $\alpha_i$, $i=1,\dots,n$, in some compact domain $D$ in $R^n$

$$\Rightarrow \int dU\, f(U) \propto \int_D d\alpha\, |\det M|^{1/2} f(U(\alpha)) \qquad \text{[differential geometry]}$$

where the metric tensor $M_{ij}(U) = \mathrm{tr}\left(\left(U^{-1}\frac{\partial}{\partial\alpha_i}U\right)U^{-1}\left(\frac{\partial}{\partial\alpha_j}U\right)\right)$

Perhaps some examples will help (!)

$\mathcal{G} = Z_2$ $\qquad \int dU = \sum_{U=\pm 1}$

$\mathcal{G} = U(1)$ $\qquad U = e^{i\theta} \Rightarrow \int dU = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta$

$\mathcal{G} = SU(2)$

Any $SU(2)$ matrix can be written $U(a) = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix}$

with $\sum_{\mu=0}^{3} a_\mu^2 = 1$ $\qquad = a_0 + i\underline{a}\cdot\underline{\sigma}$

$$\Rightarrow \int dU = \frac{1}{2\pi^2}\int d^4a\, \delta(a^2-1)$$

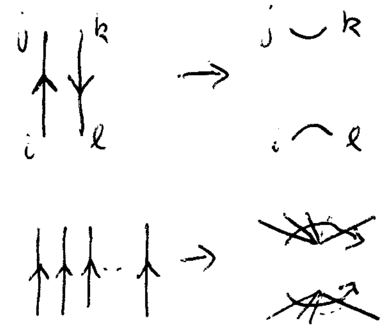
Using the above properties it is also possible to derive rules directly on matrix representations of $U$

(see Creutz ch. 8, or Eriksson et al. J. Math. Phys. <u>22</u> (1981) 2276) e.g. for $U$ in fundamental rep$^n$. of $SU(N)$

$$\int dU \, \mathbb{1} = 1 \qquad \int dU \, U_{ij} = \int dU \, U^\dagger_{ij} = 0$$

where $i, j$ are color indices running from 1 to $N$

less trivially: $$\int dU \, U_{ij} U^\dagger_{kl} = \frac{1}{N} \delta_{jk} \delta_{il} \quad \text{ie}$$



$$\int dU \, U_{i_1 j_1} U_{i_2 j_2} \dots U_{i_N j_N} = \frac{1}{N!} \varepsilon_{i_1 \dots i_N} \varepsilon_{j_1 \dots j_N}$$

totally antisymmetric N-tensors

We will use these rules to develop a <u>strong coupling expansion</u>

Exercise: Consider the expectation value of a single link

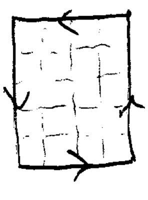
$$\langle U_\mu(x) \rangle = \frac{1}{Z} \int DU \, U_\mu(x) \exp\left( \frac{\beta}{N} \sum_{x\mu\nu} \operatorname{tr} \operatorname{Re} U_{\mu\nu}(x) \right)$$

Use the invariance property of the Haar measure to implement a change of variables so that the action $S$ does <u>not</u> depend on $U_\mu(x)$ (Hint: think about the set of links emanating from site $x$)

Hence show that $\langle U_\mu(x) \rangle \equiv 0$

This is an example of <u>Elitzur's Theorem</u>: all gauge-non invariant combinations of the $U$'s have vanishing expectation values. Only traces over closed paths of links - <u>Wilson loops</u> - can be non-vanishing

Wilson loop



$\Rightarrow$ There is no analogue of spontaneous magnetisation $\langle \sigma_i \rangle$ ie a "local order parameter" in LGT.

- One final point: the measure $\int DU$ is <u>compact</u>, and operations with it yield finite results. There is no "infinite volume of the gauge group" resulting from an attempt to integrate over non-compact variables $\int DA_\mu$. Hence there is no need for gauge fixing or the Fadeev-Popov procedure in LGT — in perturbation theory gauge fixing is needed because inappropriate integration variables are used, which only parametrise the group manifold locally, not globally.