The Deborah and Weissenberg numbers

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Since their introduction to the literature nearly fifty years ago the Deborah and Weissenberg numbers have proved invaluable for rheologists in quantifying viscoelastic effects. Despite their different origins, and the fact that they quantify *different* effects, they are frequently used as synonyms. This situation is perhaps partly explainable given that in a wide range of flows of complex fluids characterised by a relaxation time (λ) the use of simple single characteristic length (L) and velocity scales (U) does indeed result in identical definitions! In this short article we return to the original papers to highlight the different effects that the Deborah and Weissenberg numbers measure.

The Deborah number

As every good rheologist knows the Deborah number owes its name to the Prophetess Deborah who, in the Book of Judges, proclaimed "The mountains flowed before the lord"[A]. The definition is due to Reiner and we can do no better than to quote his original paper[B]

> "Deborah knew two things. First, that the mountains flow, as everything flows. But, secondly, that they flowed before the Lord, and not before man, for the simple reason that man in his short lifetime cannot see them flowing, while the time of observation of God is *infinite*. We may therefore well define a nondimensional number the Deborah number
>
> D = time of relaxation/time of observation."

[A]As Huilgol notes "Reiner quoted the Prophetess Deborah as saying "The mountains flowed before the Lord" while the King James version says "The mountains melted from before the Lord", and the New English Bible has "Mountains shook in fear before the Lord," (See Judges 5:5) [R R Huigol (1975) "On the concept of the Deborah number". Trans Soc. Rheo. 19 (2) pp297-306.]
[B]The paper by Reiner is remarkable for a number of reasons. Not the least of which is that it is a reproduction of his after-dinner speech to the Fourth International Congress on Rheology in 1962. We note, sadly, that this mode of communicating important ideas seems to have gone rather out of fashion. [M Reiner (1964) "The Deborah number". Phys. Today. 17, pp62.]

If the time of observation is long or the relaxation time of the material is short then “fluid-like” behaviour is to be expected. Conversely if the relaxation time of the material is large, or the time of observation short, then the Deborah number is high and the material behaves, for all practical purposes, as a solid. Materials which have relaxation times such that they can exhibit both fluid-like and solid-like characteristics on friendly time scales (i.e. on the order of the attention span of a human) are useful in illustrating these concepts. Silly putty is one such material; when pulled apart rapidly this material exhibits brittle fracture and when dropped onto a hard surface, not necessarily from a great height, it will bounce. Left to its own devices (well the slow deformation due to gravity) silly putty flows. For those with more patience the long-running “pitch drop” experiment[C], started in 1927 at the University of Queensland, is another material useful for illustrating the Deborah number concept.

The Deborah number (which is now commonly abbreviated to *De* rather than *D*) thus distinguishes how a particular material will behave over a given timeframe (we inherently assume that the material must be experiencing a deformation over this timeframe). Shortly after Reiner’s work, the significance of the Deborah number was discussed within a more rigorous framework by Metzner, White and Denn[1-2] and the definition slightly altered to reflect that the “time of observation” is perhaps better described as “time scale of the process” or “fluid residence time” in a given regime within a complex flow field. Although more complex definitions of *De* have been proposed[3] the simplicity of Reiner’s approach is the one that has stood (with the subtle, yet important, distinction discussed above regarding “observation” time). We thus define *De* as

$$De = \lambda / T$$

where T is a characteristic time for the deformation process and λ is still the relaxation time.

Contained within the definition of the Deborah number is the idea that it is a dimensionless measure of the rate of change of flow conditions and is therefore related to flow unsteadiness (in a material or Lagrangian sense).

[C] In this experiment pitch contained within a glass funnel slowly forms a pedant drop which eventually pinches off (liquid-like behaviour). When pitch is hit with a hammer it shatters like a brittle solid. Eight such drops have formed and pinched off since the experiment started. For those tired of watching paint dry the experiment can be viewed in real time here: http://www.smp.uq.edu.au/pitch [R Edgworth, B J Dalton and T Parnell (1984) "The Pitch Drop Experiment", Eur. J. Phys pp.198-200.]

In slowly changing or essentially steady flows, such as in fully-developed duct flows or viscometric flows like steady simple shear, the characteristic time for the deformation process is infinite and the Deborah number is therefore zero for such flows *regardless* of the relaxation time of the material. Thus the Deborah number alone is insufficient to fully characterise effects due to viscoelasticity.

The Weissenberg number

White[4] used dimensional analysis to make the equations of motion for the *steady* flow of a second order fluid dimensionless. Doing so three significant dimensionless groups arise: a group representing the ratio of inertial to viscous forces (the ubiquitous Reynolds number[5] of classical fluid mechanics),

$$Re = \rho UL / \mu$$

where is the ρ density and the μ viscosity. A group representing elastic forces to viscous forces

$$\lambda U / L$$

and a higher-order property ratio (which, in this case, is essentially the ratio of N_2 to N_1: White called it the "viscoelastic ratio" number). White interpreted the group of elastic forces to viscous forces as representing the recoverable strain in the fluid and quotes Weissenberg's paper[6] from the First International Rheological Congress held in 1948

> "As a dimensionless quantity of tensorial character, we may quote here the recoverable strain. Just as the Reynolds Number coordinates the rheological states with respect to the similitude in the relative proportions of the forces of inertia and of internal friction, so the recoverable strain does with respect to similitude in anisotropy in the sheared states...".

As a consequence he termed the group $\lambda U / L$ the "Weissenberg number". To avoid confusion with the Weber number (*We*) the abbreviation now commonly used is *Wi*.

It is instructive here to understand why the group $\lambda U / L$ does indeed represent the ratio of elastic to viscous forces. In steady simple shear flow (SSSF) for example the dominant elastic force will be due to the first normal-stress difference, $\tau_{xx} - \tau_{yy}$, and the viscous force is simply the shear stress τ_{xy}. The simplest useful differential model for a viscoelastic fluid in common usage is probably the upper-convected Maxwell model due to Oldroyd[7] and for this model in SSSF: $N_1 = \tau_{xx} - \tau_{yy} = 2\lambda\mu\dot{\gamma}^2$ and $\tau_{xy} = \mu\dot{\gamma}$. The Weissenberg number is thus:

$$Wi = \frac{2\lambda\mu\dot{\gamma}^2}{\mu\dot{\gamma}} = 2\lambda\dot{\gamma}$$

where the characteristic deformation rate $\dot{\gamma}$ can be estimated through the characteristic length and velocity scales as U / L. Expressed in this manner the relation to "recoverable shear" is also apparent.

Dimensional analysis

Dimensional analysis is frequently used in Newtonian fluid mechanics to determine appropriate dimensionless groups for a particular problem and, as we have seen already, can also be used for viscoelastic fluid flow problems. We show here how *Wi* and *De* both arise through such an analysis for a viscoelastic fluid flow.

Buckingham Π theorem states that the number of dimensionless groups (n) that can be formed from a given set of variables (k) is

$$n = k - j$$

where j is the number of independent dimensions that appear in the k variables. Given that in viscoelastic flow problems the only additional parameter[D] is the relaxation time, and no new dimensions appear, it seems reasonable to ask why two extra non-dimensional groups arise (*De* and *Wi*).

[D]Of course many other parameters may well arise in a complete analysis of real fluids such as the solvent to total viscosity ratio, the ratio of first and second normal-stress difference, etc.

For the steady, incompressible isothermal flow of a Newtonian fluid (in the absence of a free surface, or Coriolis forces, etc.) in a geometry with a single important length scale we can write

$$R = f\,(Re)$$

where R is used to denote any chosen process variable or process result in dimensionless form (the friction factor in fully-developed smooth pipe flow for example). For unsteady flows – characterised by a given frequency ω say – an additional group arises

$$R = f\,(Re,\ \omega L\,/\,U)$$

where the dimensionless group involving frequency is called the Strouhal number (*St*). This number represents the ratio of unsteady inertial forces to steady inertial forces.

For the steady, incompressible isothermal flow of a viscoelastic fluid (in the absence of a free surface, or Coriolis forces, etc.) the addition of the fluid relaxation time leads to

$$R = f\,(Re,\ Wi).$$

As the flow is steady the dimensionless group that arises due to viscoelastic effects must be *Wi* for the reasons outlined above (with the caveat that the flow must be steady in a Lagrangian sense rather than Eulerian). For unsteady viscoelastic flows

$$R = f\,(Re,\ Wi,\ \omega\lambda)$$

where $\omega\lambda$ is simply the Deborah number (and ω now can be thought of in more general terms as the reciprocal of a characteristic time of the deformation process i.e. $1\,/\,T$). In arriving at this last equation we could have derived any two groups from a choice of *Wi*, *St* and *De* (note $St = De\,/\,Wi$). When there is more than one dimensionless group, dimensional analysis places no restriction on which groups we derive: for example a group representing the ratio of elastic to inertia forces is equally as valid from purely dimensional considerations ($Wi\,/\,Re$ which is called the first Elasticity number[8]). For a broad class of viscoelastic fluid flows inertia effects are usually small, either by nature (e.g. the viscous flow of melts) or by design (viscous Boger fluids used for benchmark experiments), and the effect of *Re* is usually neglected. In such cases the Strouhal number is probably also of less importance and *De* and *Wi* become the governing groups.

Conditions under which *De* is equivalent to *Wi*.

As we have discussed above there are a limited range of Lagrangian steady flows, specifically any motion with constant stretch history[3,9], where the Deborah number is zero. Clearly in such cases the Weissenberg number must be used to quantify elastic effects. Of course, it is likely that flows in complex geometries of engineering interest will never be steady in such a sense. In such geometries if one length scale determines the dynamics of the problem the definitions for *De* and *Wi* will coincide. For example in the lid-driven cavity flow[10] an appropriate residence time for the fluid will be the length of the lid divided by its velocity ($T = L / U$), and a suitable estimate for the deformation rate is the lid velocity divided by the depth of the cavity ($\dot{\gamma} = U / H$). For a square cavity ($L = H$) then $De = Wi = \lambda U / L$. When the cavity is not square *De* and *Wi* are then related through a geometric factor. In geometries where two length scales are important in determining the dynamics the two numbers can always be related through a geometric factor. Nevertheless, as we have shown, this should not lead one to assume that *for all flows* there is no distinction between the two. It is not just a question of semantics as to which to use!

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Postscript

I wrote this article in late 2009 after the IMA meeting at the suggestion of Prof. Morton Denn (Levich Institute). At around the same time Prof. John Dealy (McGill University) published a more rigorous article[11] on the same topic.

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