

Development-Length Requirements for Fully Developed Laminar Flow in Concentric Annuli

R. J. Poole

Department of Engineering,
University of Liverpool,
Brownlow Street,
Liverpool L69 3GH, UK
e-mail: robpoole@liv.ac.uk

In this technical brief we report the results of a systematic numerical investigation of developing laminar flow in axisymmetric concentric annuli over a wide range of radius ratio ($0.01 < R_i/R_o < 0.8$) and Reynolds number ($0.001 < Re < 1000$). When the annular gap is used as the characteristic length scale we find that for radius ratios greater than 0.5 the development length collapses to the channel-flow correlation. For lower values of radius ratio the wall curvature plays an increasingly important role and the development length remains a function of both radius ratio and Reynolds number. Finally we show that the use of an empirical modified length scale to normalize both the development length and the characteristic length scale in the Reynolds number collapses all of the data onto the channel-flow correlation regardless of the radius ratio. [DOI: 10.1115/1.4001694]

1 Introduction

Knowledge of the length of duct required for so-called “fully developed” conditions to occur is important from both a fundamental and a practical standpoint. Unsurprisingly, determining this length for a range of ducts, e.g., pipe, channel, and annuli, has been the subject of a great deal of attention over the past 100 years or so [1]. Perhaps more surprisingly is that only relatively recently have accurate correlations been proposed that cover the basic pipe and channel geometries for a wide range of Reynolds numbers. A detailed discussion of the inconsistencies and confusion in literature is provided by Durst et al. [1], who conducted a detailed numerical study and proposed the following nonlinear correlations for pipes:

$$X_D/D = [(0.619)^{1.6} + (0.0567 \text{ Re})^{1.6}]^{1/1.6} \quad (1)$$

and two-dimensional channels

$$X_D/h = [(0.631)^{1.6} + (0.0442 \text{ Re})^{1.6}]^{1/1.6} \quad (2)$$

where X_D is the so-called development length, D is the pipe diameter, and h is the channel height. Equations (1) and (2) are valid in the range $0 < Re < Re_{CR}$. (As is well known, with great care, transition in pipe flow can be delayed to very high Reynolds numbers [2] but for channel flows only up to $Re_{CR} \sim 1000$ based on channel half height [3,4].) Thus the situation for pipe and channel flows is now well understood and an accurate correlation is available. For annular flows, however—arguably of more practical importance than two-dimensional channel flows, which are rarely observed in engineering practice—no such accurate correlation exists and the purpose of this short technical brief is to fill this gap. The paper of Nouar et al. [5] neatly reviews literature up until that date for this problem. However, the results of Ref. [5] are

limited in both Reynolds number range ($10 < Re < 500$), and detailed results are confined to a single radius ratio.

Of course, in the limit of the annular spacing tending to zero, the annular flow configuration approaches a two-dimensional channel and so the correlation given in Eq. (2) becomes applicable. However, with decreasing radius ratio ($N = \text{inner-cylinder radius}/\text{outer-cylinder radius} = R_i/R_o$), wall curvature effects will become increasingly important and deviation from the correlation must be expected. In the current study we will show that Eq. (2) is valid for $N > 0.5$ but that, below this value of N , significant departures in the development length are observed from the channel-flow correlation.

2 Numerical Method

To compute the developing flow field within a range of concentric annuli we make use of the fact that the flow is laminar, incompressible, steady, and axisymmetric (i.e., two-dimensional). The governing equations are then those expressing conservation of mass (Eq. (3)) and momentum (Eq. (4))

$$\frac{1}{r} \frac{\partial}{\partial r}(vr) + \frac{\partial u}{\partial x} = 0 \quad (3)$$

$$\rho \left[v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x} \right] = - \frac{\partial p}{\partial x} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial x^2} \right)$$

$$\rho \left[v \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial x} \right] = - \frac{\partial p}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv) \right) + \frac{\partial^2 v}{\partial x^2} \right) \quad (4)$$

where u is the axial velocity, v is the radial velocity, ρ is the density, μ is the dynamic viscosity, and p is the pressure.

We utilize the commercial package FLUENT to solve the governing equations of conservation of mass and momentum. This code uses a finite-volume formulation (see, e.g., Ref. [6] for details). The differencing schemes used are both formally second-order in accuracy: Central differencing is used for the diffusive terms and a second-order up-winding scheme for the convective terms. Coupling of the pressure and velocity was achieved using the well-known semi-implicit method for pressure-linked equations (SIMPLE) implementation of Patankar [7]. This well-established code has been used extensively in the calculation of complex flows (see Ref. [8] for recent examples) and is adequate to model the laminar flows under consideration here.

A schematic representation of the coordinate system and computational domain is provided in Fig. 1. The concentric circular annular geometry is characterized by the ratio of the inner radius R_i to the outer radius R_o , which here we denote by the symbol N . As previously discussed in the Introduction an important length scale is also the annular spacing h . Based on initial computational results we confine our detailed simulations to four different radius ratios, which cover the various regimes, corresponding to the vanishing inner-cylinder case ($N=0.01$), a “low” ($N=0.1$), medium ($N=0.5$), and a “high” radius ratio case ($N=0.8$). As our results show that at $N=0.8$ the two-dimensional channel correlation is already met it was not felt necessary to go to a higher value. These different radius ratios are shown schematically in the right hand side of Fig. 1: The solid line corresponds to $N=0.5$ while the other radius ratios are indicated by dashed lines. At the inlet ($x=0$) we apply a uniform velocity U_B and we define the development length X_D as the axial distance required for the maximum velocity to reach 99% of its fully developed value. We use the well-known no-slip boundary condition at the wall and impose zero axial gradients at the outlet. The length of the domain is dependent on the Reynolds number and radius ratio of the flow in question ($L = f(Re, N)$): In general the domain was at least five times as long as the calculated development length. Calculations with extended domain lengths confirmed that this criterion was sufficient to allow X_D to be independent of this length.

Contributed by the Fluids Engineering Division of ASME for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received June 10, 2009; final manuscript received April 28, 2010; published online June 11, 2010. Assoc. Editor: Mark Stremmer.

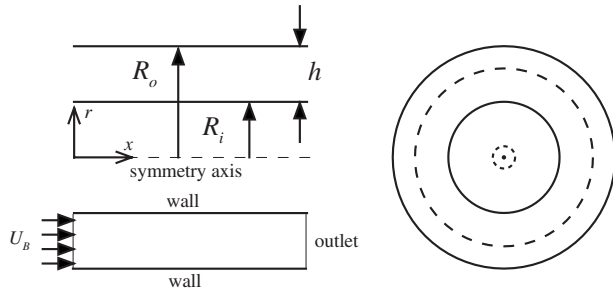


Fig. 1 Schematic of coordinate system, computational domain, and definitions including radius ratios studied (the inner solid line on the right hand side diagram corresponds to $N=0.5$ while the other radius ratios are indicated by dashed lines)

For each radius ratio a preliminary series of calculations was carried out at a low Reynolds number ($Re=0.1$) to determine a suitable mesh density and to investigate the accuracy of our simulations. In addition to the variation in X_D , to allow us to estimate this accuracy, we define a relative error

$$E = \left| \frac{u_m - U_{m,FD}}{U_{m,FD}} \right| \quad (5)$$

where u_m is the calculated maximum velocity at the outlet plane and $U_{m,FD}$ is the corresponding fully developed analytical value. The complete analytical solution for fully developed annular flow is provided in standard fluid mechanics text books and so is not unnecessarily repeated here (see Ref. [9], for example); however,

the $U_{m,FD}$ values corresponding to each of our geometries are included together with the results of our grid-dependency study in Table 1. First we note that the variation of X_D between meshes is at most about 2.1%. If we estimate the “Richardson” extrapolation value for this quantity (i.e., the value extrapolated to zero mesh size; see Ref. [10], for example) we still find that the error in our simulations, defined as $e_r = |(X_{D,MX} - X_{D,Extrap}) / X_{D,Extrap}|$ where $X_{D,M1}$ represents the development length obtained for each mesh $M1$ ($X=1$), etc., especially for meshes $M2$ and $M3$, is small ($<1.0\%$).

Based on these levels of error, and the amount of computing time required for a specific mesh density, we conducted all remaining calculations using a mesh density corresponding to mesh $M2$, which, for the case studied above, gives “errors” (both based on our E parameter and in comparison to the zero grid-size extrapolation e_r) less than 1.0%.

3 Results

A complication that arises with flow through annuli—which is absent in simple pipe and channel flows—is the different choices of length scale available with which to define Reynolds numbers and to normalize the development length. In addition to the outer radius (R_o), inner radius (R_i), and annular gap or spacing ($h=R_o - R_i$) the hydraulic diameter concept (see Ref. [9], for example) can also be employed, which here for annuli is simply $D_H=2h$. The simplest scaling is to use the outer radius: In an experimental investigation, for example, it is likely the outer cylinder would stay fixed and centerbodies of different diameters be inserted. The results of our numerical study, together with the results of Ref. [5] for comparison, are shown in Fig. 2(a). In this figure the devel-

Table 1 Mesh characteristics and development lengths for various radius ratios at $Re=0.1$ (NC=total number of grid points)

N=0.8 ($L=10h$)								
Mesh	$\Delta r/h$	$\Delta x/h$	NC	u_m/U_B	E (%) [$U_{m,FD}/U_B=1.5008$]	X_D/R_o	e_r (%)	
M1	0.004	0.008	12,500	1.4973	0.23	0.1236	2.29	
M2	0.002	0.004	50,000	1.5000	0.06	0.1257	0.66	
M3	0.001	0.002	200,000	1.5006	0.02	0.1263	0.19	
Richardson extrapolation						0.1265		
N=0.5 ($L=10h$)								
Mesh	$\Delta r/h$	$\Delta x/h$	NC	u_m/U_B	E (%) [$U_{m,FD}/U_B=1.5078$]	X_D/R_o	e_r (%)	
M1	0.01	0.02	12,500	1.5043	0.232	0.3248	2.00	
M2	0.005	0.01	50,000	1.5069	0.057	0.3282	0.97	
M3	0.0025	0.005	200,000	1.5075	0.016	0.3305	0.28	
Richardson extrapolation						0.3314		
N=0.1 ($L=10h$)								
Mesh	$\Delta r/h$	$\Delta x/h$	NC	u_m/U_B	E (%) [$U_{m,FD}/U_B=1.5673$]	X_D/R_o	e_r (%)	
M1	0.018	0.036	12,500	1.5641	0.20	0.7375	2.09	
M2	0.009	0.018	50,000	1.5664	0.06	0.7488	0.59	
M3	0.0045	0.009	200,000	1.5671	0.01	0.7520	0.17	
Richardson extrapolation						0.7532		
N=0.1 ($L=10h$)								
Mesh	$\Delta r/h$	$\Delta x/h$	NC	u_m/U_B	E (%) [$U_{m,FD}/U_B=1.6613$]	X_D/R_o	e_r (%)	
M1	0.0198	0.0396	12,500	1.6668	0.33	0.9601	0.75	
M2	0.0099	0.0198	50,000	1.6629	0.10	0.9637	0.38	
M3	0.00495	0.0099	200,000	1.6618	0.03	0.9663	0.11	
Richardson extrapolation						0.9673		

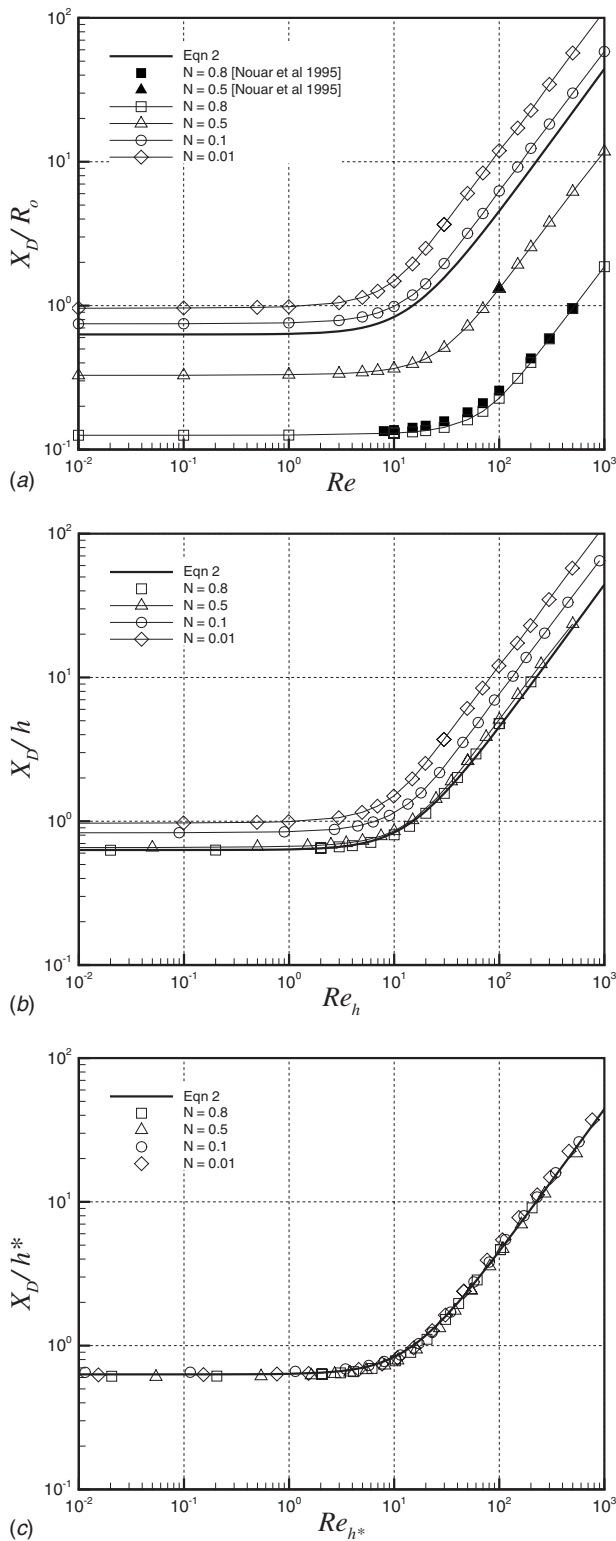


Fig. 2 Effect of radius ratio on normalized development length variation with Reynolds number (a) using R_o as the length scale, (b) using h as the length scale and (c) using h^* (Eq. (6)) as the length scale

oment length is normalized by R_o and the Reynolds number defined as $Re = \rho U_B R_o / \mu$. As was observed in the study of Durst et al., at low Re —where the flow is diffusion dominated—the development length is essentially constant (although nonzero) and increases with increasing inertia above a certain Re . Using this scal-

ing we can see that there is a significant effect of radius ratio: For example, in the low- Re limit, the development length increases by a factor of about 10 between $N=0.8$ and $N=0.01$.

As was mentioned in the Introduction, in the limit of $R_i \rightarrow R_o$ (or $h \rightarrow 0$), the annular geometry approaches the channel-flow limit and we should expect h to become the dominant length scale. Replotting the data in this manner (e.g., X_D normalized by h and the Reynolds number defined as $Re_h = \rho U_B h / \mu$) in Fig. 2(b) confirms that the data at high radius ratios ($N=0.8$) do indeed collapse to the channel-flow correlation of Durst et al. It is also clear that even for $N=0.5$ the data are sufficiently close for the correlation to be usable. Unfortunately for lower values of N the data show a marked discrepancy from the channel-flow correlation: At low Re for $N=0.01$, for example, the development length is 50% greater than that for the channel-flow case. In an attempt to collapse all of the data (i.e., for all N values) onto the channel-flow correlation we investigated the use of (twice) the distance from the outer wall to the location of the peak velocity (as the peak moves closer to the inner cylinder with increasing N) but this scaling also failed to collapse the data. A length scale based on a scaling factor such that the fully developed Poiseuille number—which depends on N (see Ref. [9] or Ref. [11])—was made to equal the channel-flow value (i.e., $f \cdot Re_h = 48$) was also unable to collapse the data (this length scale is called the “laminar equivalent diameter” [12]). Given these difficulties we choose to adopt a purely empirical approach by determining a length scale h^* such that when the low- Re development length limit is normalized by this value it collapses to the channel-flow value ($X_D/h = 0.631$). Selecting this length to normalize both the development length and the length scale in the Reynolds number collapses the data onto the channel-flow correlation, as shown in Fig. 2(c). That this length scale must also be used in Re (a plot of X_D/h^* versus Re_h does not collapse) provides confidence in this approach. This modified length scale is reasonably well fitted (the square of the correlation coefficient $R^2=0.983$) by the following expression:

$$h^*/h = -0.119 \ln(N) + 1 \quad (6)$$

which is valid in the region $0.01 < N < 1.0$. Although an empiricism, we believe that this approach represents a reasonable solution to enabling the channel-flow correlation to be used for all radius ratios.

4 Conclusions

We have reported the results of a systematic numerical investigation of developing laminar flow in axisymmetric concentric annuli over a wide range of radius ratio ($0.01 < R_i/R_o < 0.8$) and Reynolds number ($0.001 < Re < 1000$). When the annular gap is used as the length scale we find that for radius ratios greater than 0.5 the development length collapses to the channel-flow correlation. The use of a modified length scale to normalize both the development length and the characteristic length scale in the Reynolds number collapses all of the data onto the channel-flow correlation regardless of the radius ratio.

References

- [1] Durst, F., Ray, S., Unsal, B., and Bayoumi, O. A., 2005, “The Development Lengths of Laminar Pipe and Channel Flows,” *ASME J. Fluids Eng.*, **127**, pp. 1154–1160.
- [2] Darbyshire, A. G., and Mullin, T., 1995, “Transition to Turbulence in Constant-Mass-Flux Pipe Flow,” *J. Fluid Mech.*, **289**, pp. 83–114.
- [3] Li, W., Xi, L., and Graham, M. D., 2006, “Nonlinear Travelling Waves as a Framework for Understanding Turbulent Drag Reduction,” *J. Fluid Mech.*, **565**, pp. 353–362.
- [4] Escudier, M. P., Nickson, A. K., and Poole, R. J., 2009, “Turbulent Flow of Viscoelastic Shear-Thinning Liquids Through a Rectangular Duct: Quantification of Turbulence Anisotropy,” *J. Non-Newtonian Fluid Mech.*, **160**, pp. 2–10.
- [5] Nour, C., Ouldrouis, M., Salem, A., and Legrand, J., 1995, “Developing Laminar Flow in the Entrance Region of Annuli—Review and Extension of Standard Resolution Methods for the Hydrodynamic Problem,” *Int. J. Eng. Sci.*, **33**, pp. 1517–1534.

- [6] Versteeg, H. K., and Malalasekera, W., 1995, *An Introduction to Computational Fluid Dynamics. The Finite Volume Method*, Longman, Harlow, UK.
- [7] Patankar, S. V., 1980, *Numerical Heat Transfer and Fluid Flow*, Hemisphere, Washington, DC.
- [8] Poole, R. J., and Ridley, B. S., 2007, "Development Length Requirements for Fully-Developed Laminar Pipe Flow of Inelastic Non-Newtonian Liquids," *ASME J. Fluids Eng.*, **129**, pp. 1281–1287.
- [9] White, F. M., 1991, *Viscous Fluid Flow*, 2nd ed., McGraw Hill, New York.
- [10] Ferziger, J. J., and Peric, M., 2001, *Computational Methods for Fluid Dynamics*, Springer, New York.
- [11] Shah, R. K., and London, A. L., 1978, *Laminar Flow Forced Convection in Ducts*, Academic, New York.
- [12] Jones, O. C., and Leung, J. C. M., 1981, "An Improvement in the Calculation of Turbulent Friction in Smooth Concentric Annuli," *ASME J. Fluids Eng.*, **103**, pp. 615–623.