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Development-Length Requirements for Fully Developed Laminar Pipe Flow of Inelastic Non-Newtonian Liquids

In the current study, we report the results of a detailed and systematic numerical investigation of developing pipe flow of inelastic non-Newtonian fluids obeying the power-law model. We are able to demonstrate that a judicious choice of the Reynolds number allows the development length at high Reynolds number to collapse onto a single curve (i.e., independent of the power-law index n). Moreover, at low Reynolds numbers, we show that the development length is, in contrast to existing results in the literature, a function of power-law index. Using a simple modification to the recently proposed correlation for Newtonian fluid flows (Durst, F. et al., 2005, "The Development Lengths of Laminar Pipe and Channel Flows," J. Fluids Eng., ***127****, pp. 1154–1160) to account for this low Re behavior, we propose a unified correlation for* X_D/D*, which is valid in the range* $0.4<n<1.5$ *and* $0<\mathrm{Re}<1000$*.* [DOI: 10.1115/1.2776969]

1 Introduction

The importance of knowing the length of pipe required for laminar Newtonian pipe flow to fully develop, i.e., for the velocity profile to become nonvarying in the axial direction, has long been recognized. Not only is this "development length" of great practical use, in the design of pipe flow systems, for example, but it is also important for scientists and engineers studying such flows and their transition to turbulence [1].

This importance is reflected in the number of studies which have attempted to provide a definitive relationship between the nondimensional entrance length (X_D/D) and the Reynolds number in a functional form $X_D/D=C_1\mathrm{Re}$. Numerous papers have used either analytical [2–4], numerical [5–7], or experimental [8,9] means to determine $X_D=f(\mathrm{Re})$. This extensive literature is succinctly described in the recent paper of Durst et al. [10] who also point out that virtually all of these studies incorrectly provide the variation of X_D as being of the form $X_D/D=C_1\mathrm{Re}$. Such a relationship incorrectly implies that in the creeping-flow limit (i.e., $\mathrm{Re}\rightarrow 0$), the flow will instantaneously develop. In fact, at low Reynolds number, diffusion plays an important role and the correct functional form of the relationship between the development length and Re should be

$$X_D/D = C_0 + C_1\mathrm{Re} \tag{1}$$

To address the inconsistencies and confusion in the literature, Durst et al. [10] conducted a detailed numerical study and proposed the following nonlinear correlation:

$$X_D/D = [(0.619)^{1.6} + (0.0567\mathrm{Re})^{1.6}]^{1/1.6} \tag{2}$$

which is valid in the range $0<\mathrm{Re}<\infty$ (provided, of course, that the flow remains laminar. As is well known, with great care, transition in pipe flow can be delayed to very high Reynolds numbers [1]). Thus, the situation for Newtonian fluids is finally well understood and an accurate correlation is now available. For non-Newtonian fluid flows, in contrast to the situation for Newtonian fluid flows, the literature is considerably scarcer although by no means less contradictory. In Table 1, we summarize most of the previous investigations using the "power-law" model in this area. Much as Durst et al. [10] observed for Newtonian fluid flows, nearly all of these previous studies predict a relationship of the form

$$X_D/D = C(\mathrm{Re}) \tag{3}$$

where $C=f(n)$ and n is the power-law index. Such correlations clearly neglect the role played by diffusion, which becomes increasingly important with decreasing Reynolds number. Based on our own numerical results, described in detail in this paper, a correlation of the form of Eq. (3) is probably only valid for Reynolds numbers greater than about 20. Moreover, there are considerable differences for even this "high Reynolds number" estimation in the Newtonian limit (i.e., $n=1$). Only three studies are within ±10% of the robust value of 0.0567Re determined recently in the numerical study of Ref. [10]: Collins and Schowalter [3], 0.061Re; Mehrota and Patience [14], 0.056Re; and Ookawara et al. [15], ~0.0575Re. This latter paper (i.e., Ref. [15]) is the only study in the literature that proposes both a correlation of the correct form and is in reasonable (maximum 5.8% error) agreement with the correlation of Durst et al. [10] in the Newtonian limit. However, the correlation proposed by Ookawara et al. [15] predicts that, in the creeping-flow limit, the development length is independent of the power-law index n and equal in magnitude to the Newtonian development length. Such a result seems surprising given the nonlinearity that is retained in the governing equations through the power-law equation (in contrast to the corresponding equations for creeping Newtonian flow, which are, of course, linear).

In the current study, we conduct a detailed and systematic numerical investigation of developing pipe flow for inelastic non-Newtonian fluids obeying the power-law model. We show that a judicious choice of the Reynolds number allows the development length at high Re to collapse onto a single curve (i.e., independent of n). Moreover, at low Re, we show that the development length is, in contrast to the results of Ookawara et al., a function of power-law index. Using a simple modification to the correlation proposed by Durst et al., to account for this low Re behavior, we propose a unified correlation for X_D/D, which is valid in the range $0.4<n<1.5$ and $0<\mathrm{Re}<1000$.

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Table 1 Summary of previous investigations of development-length requirements for non-Newtonian power-law pipe flows

Author	Method	Parameter range	Re definition	Re range	$X_D=f(\mathrm{Re})$	Newtonian prediction
Collins and Schowalter [3]	A	$0<n<1$	Re_{CS}	No range provided	$X_D/D=C(\mathrm{Re})$ where $C=f(n)$	$X_D/D=0.061(\mathrm{Re})$
Mashelkar [11]	A	$0<n<1$	Re_{CS}	"High Re"	$X_D/D=C(\mathrm{Re})$ where $C=f(n)$	$X_D/D=0.049(\mathrm{Re})$
Soto and Shah [12]	N	$n=0.5$, 0.75 and 1.5	Re_{CS}	No range provided	$X_D/D=(0.15-0.085n)\mathrm{Re}^{a}$	$X_D/D=0.065(\mathrm{Re})$
Matros and Nowak [13]	A	No limit provided	Re_{MR}	No range provided	$X_D/D=\mathrm{Re}\left\{0.0865\left[\frac{2(n+1)}{3n+1}\right]^{-2}\right\}$	$X_D/D=0.0865(\mathrm{Re})$
Mehrota and Patience [14]	N	$0.6<n<1.5$	Re_{MR}	>200	$X_D/D=0.056(\mathrm{Re})$	
Ookawara et al. [15]	N	No limit provided	Consult ref	<50	$X_D/D=\sqrt{((0.655)^2+(0.0575)^2(\mathrm{Re})^2)}$	
Gupta [16]	A	$0.3<n<2.0$	Re_{MR}	No range provided	$X_D/D=C(\mathrm{Re})$ where $C=f(n)$	$X_D/D=0.04(\mathrm{Re})$
Chebbi [17]	A	$0<n<1.5$	Re_{CS}	No range provided	$X_D/D=C(\mathrm{Re})$ where $C=f(n)$	$X_D/D=0.09(\mathrm{Re})$

[a]Extracted by the current authors from graphical data

2 Numerical Method

To compute the developing flow field within a pipe for inelastic non-Newtonian liquids, we make use of the fact that the flow is laminar, incompressible, steady, and axisymmetric (i.e., two dimensional). The governing equations are then those expressing conservation of mass (Eq. (4)) and momentum (Eq. (5)) in combination with a suitable rheological constitutive equation,

$$\frac{\partial u_i}{\partial x_i}=0 \tag{4}$$

$$\rho\left[\frac{\partial(u_i u_j)}{\partial x_i}\right]=\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\eta(\dot{\gamma})\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] \tag{5}$$

For reasons of simplicity, we choose to use a purely viscous generalized Newtonian fluid (GNF) based on the power-law model, which gives the following "viscosity function":

$$\eta(\dot{\gamma})=k\dot{\gamma}^{n-1} \tag{6}$$

where the shear rate $\dot{\gamma}$ is related to the second invariant of the rate of deformation tensor (D_{ij}) by

$$\dot{\gamma}=\sqrt{2D_{ij}D_{ij}} \quad \text{where } D_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{7}$$

Equations (4)–(7) are solved using the commercial package FLUENT (Version 6.2). This well-established code has been used extensively in the calculation of complex flows (see Refs. [18–22] for recent examples) and is adequate to model the inelastic laminar flows under consideration here. The differencing schemes used are both formally second order in accuracy: central differencing is used for the diffusive terms and a second-order upwinding scheme for the convective terms. Some limited calculations were also carried out using a theoretically third-order quadratic upstream interpolation for convective kinematics (QUICK) type scheme for the convective terms to ascertain the effect of discretization scheme on the accuracy of our results. Coupling of the pressure and velocity was achieved using the well-known semi-implicit method for pressure-linked equations (SIMPLE) implementation of Patankar [23].

Double precision (14 d.p.) was used for all the calculations so that round-off errors are negligible. The iterations were stopped whenever the scaled residuals (see Ref. [24]) for the solutions for the two components of velocity and the continuity equation approached an asymptotic value; in general, the scaled residuals were observed to reach a level between 10^{-12} and 10^{-15}.

A schematic representation of the computational domain is given in Fig. 1. At inlet ($x=0$), we apply a uniform velocity U_B and we define the development length X_D as the axial distance required for the centerline velocity to reach 99% of its fully developed value. We use the well-known no-slip boundary condition at the wall and impose zero axial gradients at the outlet. The length of the domain is dependent on the Reynolds number of the flow in question ($L=f(\mathrm{Re})$); in general, the domain was at least five times as long as the calculated development length. Calculations with extended domain lengths confirmed that this criterion was sufficient to allow X_D to be independent of this length.

A preliminary series of calculations was carried out for a Newtonian fluid, at a moderate Reynolds number ($\mathrm{Re}=10$), with 10×20, 20×40, 40×80, 80×160, and 160×320 quadrilateral cells of constant dimension, $\Delta x=2\Delta r$, to determine a suitable mesh density and to investigate the accuracy of our simulations. In addition to the variation in X_D, to allow us to estimate this accuracy, we define a relative error

$$E=\frac{u_c-U_{C,\mathrm{FD}}}{U_{C,\mathrm{FD}}} \tag{8}$$

where u_c is the calculated centerline velocity at the outlet plane and $U_{C,\mathrm{FD}}$ is the corresponding fully developed analytical value. The analytical solution for the fully developed pipe flow of a power-law fluid is well known (see Ref. [25], for example) and is given by

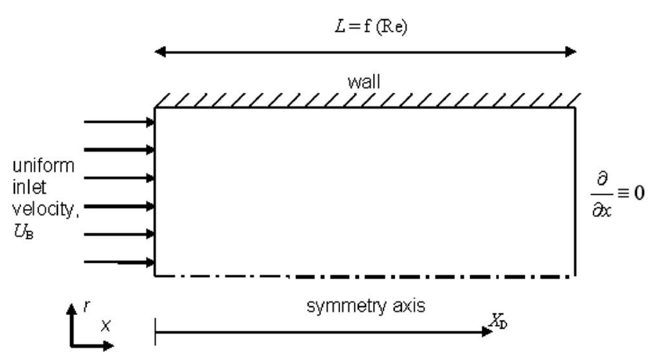

Fig. 1 Schematic of computational domain and boundary conditions

Table 2 Mesh characteristics and development lengths for a Newtonian fluid Re=10 using either a second-order upwind or QUICK-type discretization scheme

Mesh	$\Delta r/R$	$\Delta x/R$	NC	u_c	Second order upwind E (%)	X_D	e_r (%)
M1	0.1	0.2	200	1.980	0.9821	0.8897	1.81
M2	0.05	0.1	800	1.995	0.2472	0.8789	0.52
M3	0.025	0.05	3200	1.999	0.06256	0.8747	0.15
M4	0.0125	0.025	12800	2.000	0.01678	0.8734	0.04
M5	0.00625	0.0125	51200	2.000	0.00534	0.8730	0.02
Richardson extrapolation						0.8729	
					QUICK		
M1	0.1	0.2	200	1.980	1.026	0.8946	1.71
M2	0.05	0.1	800	1.995	0.2611	0.8768	0.53
M3	0.025	0.05	3200	1.999	0.06704	0.8749	0.16
M4	0.0125	0.025	12800	2.000	0.01778	0.8735	0.05
M5	0.00625	0.0125	51200	2.000	0.00534	0.8730	0.02
Richardson extrapolation						0.8728	

$$\frac{u}{U_B} = \frac{(1+3n)R^{-(1+n)/n}}{(1+n)}(R^{(1+n)/n} - r^{(1+n)/n}) \tag{9}$$

Thus, the centerline velocity is simply $U_{C,\mathrm{FD}}=[(1+3n)/(1+n)]U_B$ and the velocity profile becomes increasingly flat with increasing degree of shear thinning index (i.e., decreasing n).

The results of our grid-dependency study are tabulated in Table 2. Firstly, we note that the variation of X_D between meshes is at most about 2%. Secondly, fitting these points to an equation of the form $a(\Delta r)^p+b$ allows us to estimate the order of accuracy (p) of our simulations and the mesh-independent (extrapolated) X_D value, b [26]. Using this order to estimate the "Richardson" extrapolation value for this quantity (i.e., the value extrapolated to zero mesh size), shown in Table 2, we still find that the error in our simulations, defined as $e_r=(X_{D,\mathrm{extrap}}-X_{MX})/X_{D,\mathrm{extrap}}$, especially for meshes M4 and M5, is exceedingly small ($<0.05\%$). Also shown in Table 2 are the corresponding results obtained using a QUICK type of discretization scheme for the convective fluxes. With increasing mesh refinement, differences between the two schemes become increasingly slight and reassuringly the extrapolated values for both schemes agree to better than 0.005%. Based on these levels of error, and the amount of computing time required for a specific mesh density, we conducted all remaining calculations using a mesh density corresponding to mesh M4 which, for the case studied above, gives "errors" (both based on our E parameter and in comparison with the zero grid-size extrapolation e_r) less than 0.05%. Using this mesh density (which at the highest Re studied, and therefore the longest domain lengths, led to grids with 640,000 cells) resulted in very accurate simulations for the whole range of Reynolds numbers and power-law indices studied. Our E parameter, for example, which essentially measures the difference between the fully developed calculation and the corresponding analytical solution, although exhibiting a slight dependency on n, was always less than 0.03%.

3 Validation of Newtonian Results

In Fig. 2, we plot the variation of the development length with Reynolds number for our Newtonian simulations. Also shown is the nonlinear correlation proposed by Durst et al. [10], i.e., Eq. (2). As can be seen, there is good agreement between the current simulations and the correlation. The maximum percentage difference between our data and the correlation proposed by Durst et al. [10] is about 4.5%. Such a level of discrepancy is slightly greater than the error between the original data of Durst et al. [10] and their correlation (3%).

4 Power-Law Model Results

4.1 Definition of Re. As discussed in detail by Chhabra and Richardson [27], as a consequence of their variable viscosity, one of the intrinsic difficulties with analyzing flows of non-Newtonian liquids is in the correct definition of a Reynolds number (i.e., the ratio of inertia to viscous forces within the flow). Perhaps, the most straightforward Reynolds number is that based on a characteristic shear rate $\dot{\gamma}=U_B/D$, which here we call the Collins–Schowalter Reynolds number

$$\mathrm{Re}_{\mathrm{CS}} = \frac{\rho U_B^{2-n}D^n}{K} \tag{10}$$

Such a Reynolds number also arises naturally from a simple dimensional analysis of the problem. Although the simplicity of Eq. (10) is appealing, it is well known that it is not always the most appropriate definition. For example, in turbulent pipe flows [28] or in pipe flows with abrupt changes in cross-sectional area [29], the viscosity at the wall is usually more useful,

$$\mathrm{Re} = \frac{\rho U_B D}{\mu_{\mathrm{wall}}} \tag{11}$$

This Reynolds number has the advantage that it is physically based on some quantity within the flow but has the disadvantage that it requires a detailed knowledge of the flow field and cannot be easily estimated in a complex flow. In a developing flow field, such as the current case, it also has the disadvantage that it will vary with axial location until the flow is fully developed. Nevertheless, if we consider the fully developed region in our flow, it is relatively straightforward to show that $\dot{\gamma}_{\mathrm{wall}}=(2(1+3n)/n)\times(U_B/D)$ and therefore that

$$\mathrm{Re}_{\mathrm{wall}} = \frac{\rho U_B^{2-n}D^n}{K}\left(\frac{n}{2+6n}\right)^{n-1} \tag{12}$$

Alternatively, one may take a different approach and select a generalization of the Reynolds number such that the data under investigation collapse in some manner. Metzner and Reed [30] used such an approach to correlate the pressure drop data required to drive the fully developed non-Newtonian flow in a pipe. They defined a Reynolds number

$$\mathrm{Re}_{\mathrm{MR}} = \frac{\rho U_B^{2-n}D^n}{K}8\left(\frac{n}{6n+2}\right)^n \tag{13}$$

so that $f=16/\mathrm{Re}_{\mathrm{MR}}$ in laminar flow. An identical definition of this Reynolds number was also derived by Bird [31] using an elegant

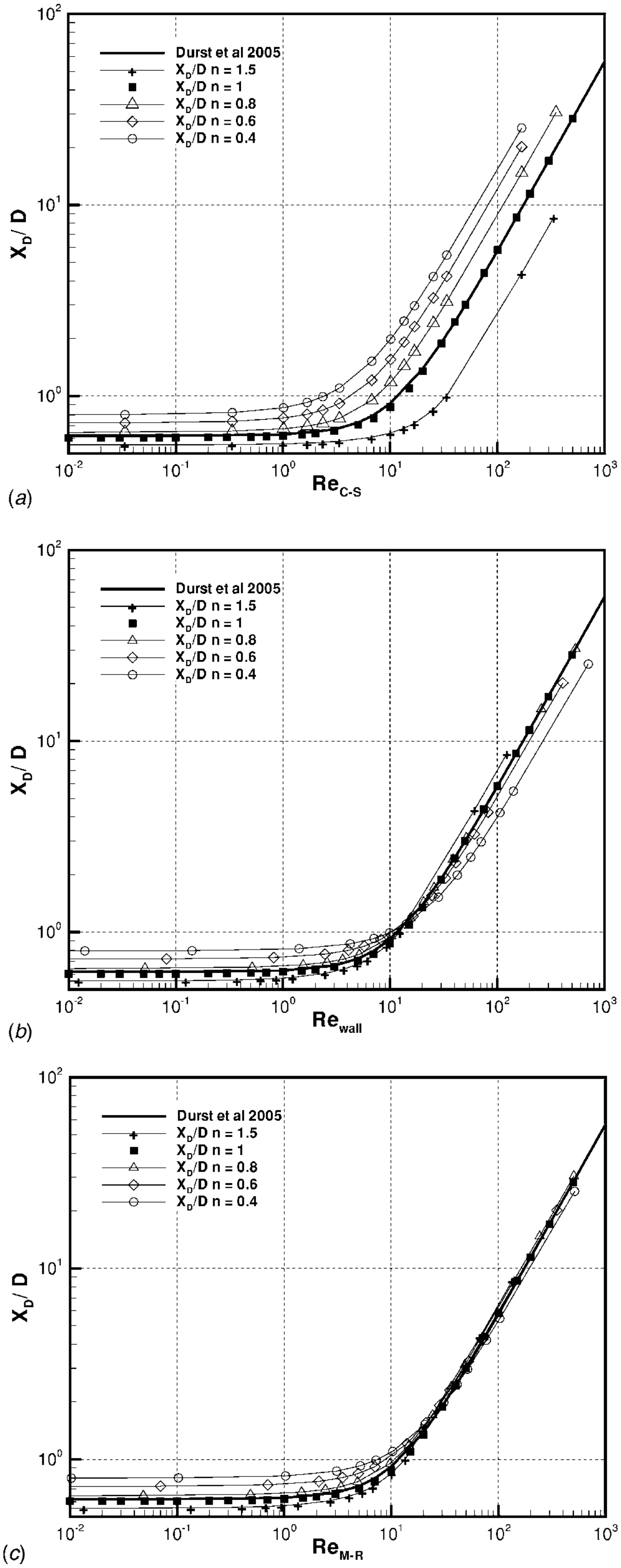

Fig. 2 Variation of development length for Newtonian and power-law fluids versus (*a*) Reynolds number based on Collins and Schowalter [3], (*b*) Re based on wall viscosity in fully-developed flow, and (*c*) Re based on definition of Metzner and Reed [30]

dimensional analysis approach. Finally, it is worth noting that all of these definitions are inter-related; thus,

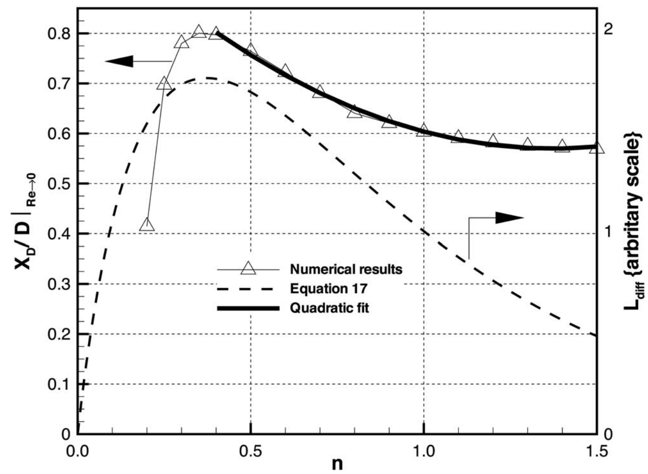

Fig. 3 Variation of creeping-flow ($Re_{CS}=0.001$) development length with power-law index

$$Re_{MR} = 8\left(\frac{n}{6n+2}\right)^n Re_{CS} \tag{14}$$

$$Re_{wall} = \left(\frac{n}{6n+2}\right)^{n-1} Re_{CS} \tag{15}$$

4.2 Discussion of Power-Law Simulations. The development length variation for the power-law model is shown for the three different definitions of the Reynolds number, Re_{CS}, Re_{wall}, and Re_{MR}, in Figs. 2(*a*)–2(*c*). Depending on the Re definition used, different conclusions can be drawn regarding the effect of power-law index on X_D. Using a Reynolds number based on a "characteristic" shear rate, Re_{CS}, a monotonic progressive increase in development length is observed for decreasing power-law index, i.e., X_D increases with increasing shear thinning ($n<1$) and decreases with shear thickening ($n>1$). Below a certain Reynolds number, $Re_{CS}\approx 1$, the flow is essentially governed by diffusion and, for a given value of n, the development length is approximately constant. At higher Re, the slope (i.e., $d[\log(X_D/D)]/d[\log(Re_{CS})]$) also appears to be approximately constant. Plotting the data using a Re based on the wall viscosity in the fully developed region, Re_{wall} (Fig. 2(*b*)), reveals a different overall trend; although at low Re, the trend is the same, at a critical Reynolds number ($Re_{wall}\approx 15$), there is a crossover past which the effect of power-law index on development length is reversed compared to the trend observed with Re_{CS}.

If we plot the data using the generalized Reynolds number of Metzner and Reed [30], we see that, at higher Reynolds number at least, the development lengths "collapse" onto the Newtonian curve. At high Reynolds numbers, our results are thus in practical agreement with the most recent numerical simulations in the literature (Mehrota and Patience [14] and Ookawara et al. [15]). At lower Re, where the development length is determined by diffusion, the constant development length is a function of power-law index. Such dependency is previously unreported in the literature, although most previous studies neglected this low-Re region.

We plot the variation of this constant creeping-flow development length with power-law index in Fig. 3. Further simulations were conducted in this low-Re regime (at $Re_{CS}=0.001$) to map out this variation in detail. Also included in Fig. 3 is a quadratic fit to these data over the range of power-law index for which we have conducted detailed simulations (i.e., $0.4<n<1.5$). With increasing shear thinning, i.e., progressively decreasing n, iterative convergence became increasingly time consuming and below $n=0.2$ we could no longer obtain convergence. A maximum development

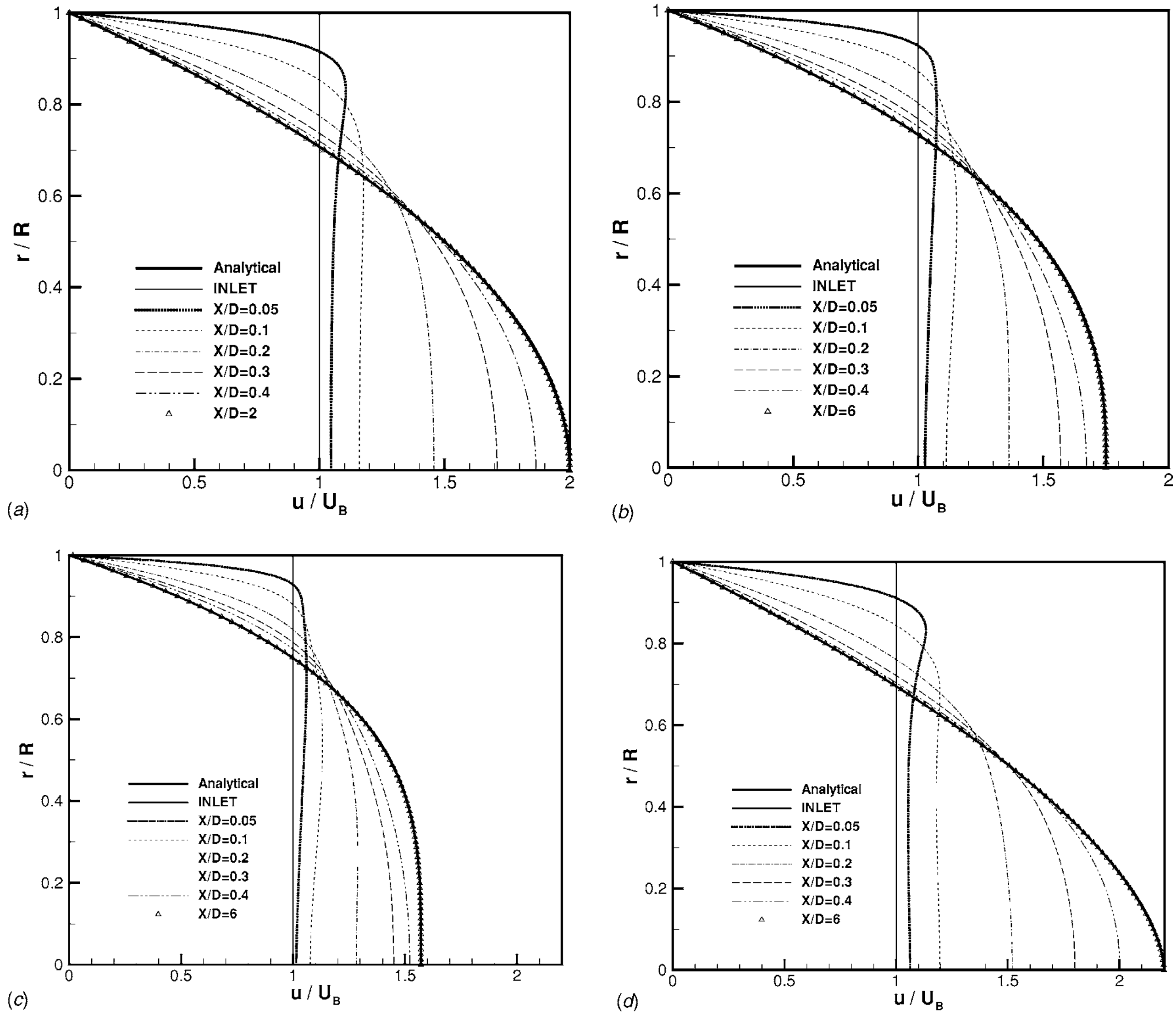

Fig. 4 Creeping-flow (Re_{CS}=0.001) axial velocity development at various axial locations for (*a*) Newtonian fluid, (*b*) *n*=0.6, (*c*) *n*=0.4, and (*d*) *n*=1.5

length occurs for $n \approx 0.35$; for fluids more shear thinning than this, the fully developed velocity profile becomes increasingly flat until the limiting behavior of $n=0$ the velocity profile is itself uniform and "instantaneously" fully developed. A possible explanation for the complex variation of the development length with power-law index is that there are two "competing" effects. In this creeping-flow regime, the development length is essentially diffusion dominated and so with decreasing n, it is plausible that the diffusion time will increase and so you might expect the development length to increase. However, with increased shear thinning, the fully developed velocity profile becomes increasingly uniform and so less rearrangement of the uniform inlet velocity profile needs to occur before it attains its fully developed shape. Using a simple scaling argument, it is possible to show that a characteristic diffusion time must be inversely proportional to the "effective" viscosity (at least in the range $0<n<2$),

$$t_{\text{diff}} \propto \frac{C^*}{\mu_{\text{eff}}} \tag{16}$$

where $\mu_{\text{eff}}=(1/8)K[(6n+2)/n]^n$ (i.e., from the Metzner-Reed Reynolds number) and C^* is an order-one constant, which has the units s^{n-1} kg/m. If we also define a characteristic velocity scale, to account for the fact that the fully developed velocity profile becomes increasingly flat with decreasing n, as the difference between the centerline fully developed value and the bulk velocity (i.e., $U_{C,\text{FD}}-U_B$), the development length should be of the form

$$L_{\text{diff}} \propto U^* t_{\text{diff}} \propto (U_{C,\text{FD}}-U_B)\frac{C^*}{\mu_{\text{eff}}} \tag{17}$$

The variation of this length scale with power-law index is included in Fig. 3. Although such a simple scaling argument based on bulk variables is unable to completely capture the full complexity of the development-length variation, it does correctly predict the power-law index at which the development length attains a maximum and it also predicts the form of the variation above and below this n value.

To further understand this complex low-Re behavior, in Fig. 4 we show the axial velocity distribution at various axial locations together with the fully developed analytical solution and the corresponding, virtually indistinguishable, fully developed numerical solution. As has been observed previously for the Newtonian case (Durst et al. (2005)), close to the uniform velocity inlet, a significant off-centerline velocity overshoot is apparent for all fluids, although this peak is enhanced for the shear-thickening fluid ($n=1.5$) and increasingly diminished with decreasing power-law in-

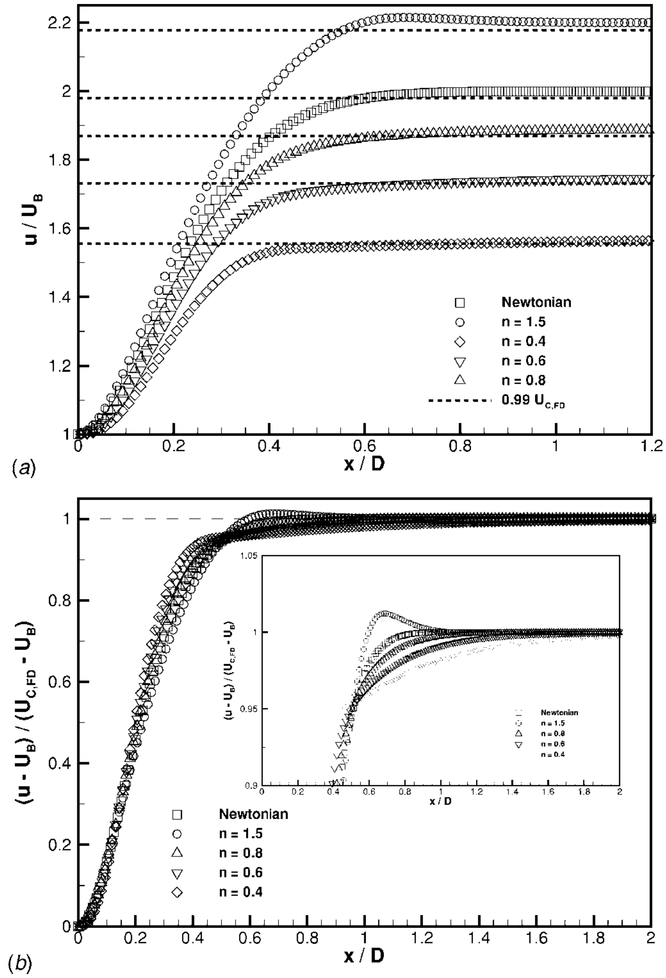

Fig. 5 Variation of centerline velocity for creeping-flow cases ($Re_{CS}=0.001$): (*a*) normalized by bulk velocity and (*b*) scaled to remove influence of flattened velocity profile

dex. As a consequence of the flattening of the velocity profiles, which accompanies shear thinning, it is difficult to differentiate this effect from any development-length effects. Therefore, in Fig. 5(*a*), we plot the centerline velocity variation, and in Fig. 5(*b*), we plot the same velocity data but scaled such that it varies between 0 and 1: 0 corresponds to the bulk uniform inlet velocity and 1 to the fully developed centerline value. Rescaling the data, as in Fig. 5(*b*), highlights some interesting effects. Close to inlet, $x/D < 0.3$, the centerline velocities collapse onto a single curve for all values of n. Downstream of $x/D > 0.5$, the rate of change of the centerline axial velocity decreases with decreasing n, at least in the range $0.4 < n < 0.8$, in agreement with the observed increase in X_D for increasing amounts of shear thinning (cf. Fig. 2). For the shear-thickening fluid, $n=1.5$, the centerline velocity initially overshoots its fully developed value. Thus, although using our definition of X_D, the shear-thickening fluids appear to develop quicker; in fact, if we were to refine our definition of X_D, perhaps to the more robust "length at which the centerline velocity attains a monotonically varying value within 1% of its fully developed value," this trend would be altered.

Finally, if we use the polynomial fit to the variation of creeping-flow development length with power-law index (shown in Fig. 3), we can make a simple modification to the correlation of Durst et al. to account for shear-thinning and mildly shear-thickening effects,

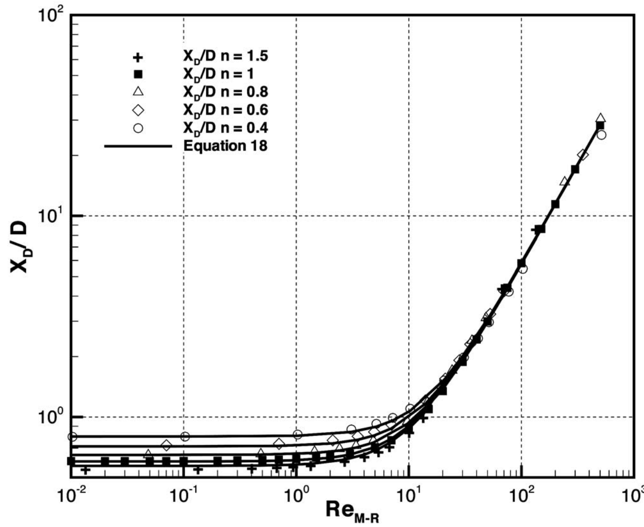

Fig. 6 Variation of development length for Newtonian and power-law fluids versus Re_{MR} together with universal correlation based on a modification to correlation of Durst et al. [10]

$$X_D/D = [(0.246n^2 - 0.675n + 1.03)^{1.6} + (0.0567\mathrm{Re}_{MR})^{1.6}]^{1/1.6} \tag{18}$$

This equation is only valid for power-law indices in the range $0.4 < n < 1.5$. The correlation is shown in Fig. 6 together with the numerical data for the various levels of power-law index. In general, the agreement of Eq. (18) to the data is better than 5% except at the highest Reynolds numbers where, especially for $n=0.4$ and 1.5, the agreement deteriorates somewhat (maximum 15% difference).

5 Conclusions

We have reported the results of a detailed and systematic numerical investigation of developing pipe flow of inelastic non-Newtonian fluids obeying the power-law model. Use of the Reynolds number developed by Metzner and Reed [30] allows the development length at high Reynolds number to collapse onto a single curve (i.e., independent of the power-law index n). Moreover, at low Reynolds numbers, the development length is, in marked contrast to existing results in the literature, a function of power-law index. Using a simple modification to the recently proposed correlation for Newtonian fluid flows [10] to account for this low-Re behavior, we have proposed a unified correlation for X_D/D, which is valid in the range $0.4 < n < 1.5$ and $0 < \mathrm{Re} < 1000$.

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