Optimal switching between two spectrally negative Lévy processes to minimise ruin probability

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Abstract

We consider an optimal underwriting problem where given two insurance portfolios that generate cash flows according to two spectrally negative Lévy processes of bounded variation X and Y, one has to underwrite adaptively a convex combination of the two such that the probability of ruin occurring in the combined portfolio is minimised. This optimal underwriting problem boils down to an optimal switching problem where one has to decide, based on the available capital at a given time, whether to go for mode X or for mode Y at that time. The 1-switch-level strategy with parameter b in $[0, \infty]$ is the strategy where one switches from one mode to the other only at times when the capital goes above or below the level b. We find a set of sufficient conditions on the two Lévy measures such that an optimal strategy is formed by a 1-switch-level strategy, which covers in particular the case where the hazard rates of the two Lévy measures are decreasing and ordered. An interesting tool in the analysis is a new monotonicity property regarding quasi-convexity for renewal equations.

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1 Problem statement

By a spectrally negative Lévy process of bounded variation we mean a real-valued process with stationary and independent increments that has no upward jumps and sample paths that are of bounded variation and non-monotone. For such a process $Z = \{Z_t\}_{t\geq 0}$, we can write

$$Z_t = c_Z t - \int_0^t \int_0^\infty z N_Z(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0, \tag{1}$$

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where $c_Z > 0$ denotes the drift and $N_Z(\mathrm{d}s, \mathrm{d}z)$ is a Poisson point process on $[0,\infty)^2$ with intensity measure $\mathrm{d}t\Pi_Z(\mathrm{d}z)$ where the Lévy measure denoted by Π_Z is a measure on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge z)\Pi_Z(\mathrm{d}z) < \infty$. By applying Tonelli to $\int_0^1 \int_0^z 1\mathrm{d}u\Pi_Z(\mathrm{d}z)$, one can easily show that

$$\int_0^1 \Pi_Z(z,\infty)\mathrm{d}z = \int_0^\infty (1\wedge z)\Pi_Z(\mathrm{d}z) < \infty. \tag{2}$$

The notation c_Z, N_Z and Π_Z will be used throughout the paper for Z a spectrally negative Lévy process of bounded variation.

In an insurance context Z as in (1) represents the surplus/capital over time of a portfolio of insurance risks with c_Z being the constant premium rate per unit of time and the point process $N_Z(\mathrm{d}t, \mathrm{d}x)$ standing for the number of claims of size $\mathrm{d}x$ appearing in the time period $\mathrm{d}t$. It is then said that ruin occurs in the portfolio if the process Z ever becomes strictly negative. In one sentence our optimal control problem of interest can be informally described as follows: given two such insurance portfolios how to choose adaptively a convex combination of the two such that the probability of ruin occurring in the combined portfolio is minimised. In particular one is allowed to hold/underwrite a proportion $q \in (0,1)$ of a given portfolio Z. What we mean by this is that then one collects the premium rate qc_Z and in return has to cover fully an incoming claim of N_Z with probability q and none of it with probability $1-q$. The latter is different from covering a proportion q of the size of each incoming claim; such a feature appears when considering proportional reinsurance (and is also typical in portfolio selection problems from the area of mathematical finance that involve Lévy processes) and gives rise to quite different optimal control problems than the optimal underwriting problem considered here.

Next we give a rigorous formulation of our control problem which is consistent with the above description. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$ be two spectrally negative Lévy processes of bounded variation such that the pair (X, Y) forms a bivariate Lévy process. Note that the latter condition is satisfied if X and Y are independent but we also want to allow for dependency. We can write, for $t \geq 0$,

$$\begin{aligned} X_t =& c_X t - \int_0^t \int_{[0,\infty)^2} xN(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y), \\ Y_t =& c_Y t - \int_0^t \int_{[0,\infty)^2} yN(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y), \end{aligned} \tag{3}$$

where $N(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y)$ is a Poisson point process on $[0,\infty)^3$ with intensity measure $\mathrm{d}t\Pi(\mathrm{d}x, \mathrm{d}y)$ where the Lévy measure Π is a measure on $[0,\infty)^2$ satisfying $\Pi(\{0\},\{0\}) = 0$, $\Pi(\mathrm{d}x, [0,\infty)) = \Pi_X(\mathrm{d}x)$ and $\Pi([0,\infty), \mathrm{d}y) = \Pi_Y(\mathrm{d}y)$. To each point in N we attach an independent and uniformly on $[0,1]$ distributed random variable. The resulting marked point process $\widetilde{N}$ is then a Poisson point process on $[0,\infty)^3 \times [0,1]$ with intensity measure $\mathrm{d}t\Pi(\mathrm{d}x, \mathrm{d}y)\mathrm{d}u$, see Section 5.2 in [8]. This marking of the point process by uniform random variables will be used to introduce a dependent thinning of N in order to determine which claims are covered. For $t \geq 0$, we denote by $\mathcal{F}_t$ the smallest σ-algebra such that the random variable $\widetilde{N}(A_1, A_2, A_3, A_4)$ is measurable for any $A_1 \in \mathcal{B}([0,t])$, $A_2, A_3 \in \mathcal{B}([0,\infty))$, $A_4 \in \mathcal{B}([0,1])$, where $\mathcal{B}(A)$ denotes the Borel σ-algebra of an interval A. A control $Q = \{Q_t\}_{t\geq 0}$ is defined to be an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process with càglàd sample paths and taking values in $[0,1]$. Given a control Q and a starting point $x \geq 0$, the controlled process $U^Q = \{U_t^Q\}_{t\geq 0}$ is defined as

$$U_t^Q = x + \int_0^t (Q_s c_X + (1-Q_s)c_Y)\mathrm{d}s - \int_0^t \int_{[0,\infty)^2} \int_0^1 \left(x\mathbf{1}_{\{u\leq Q_s\}} + y\mathbf{1}_{\{u>Q_s\}}\right) \widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y, \mathrm{d}u). \tag{4}$$

Note that U_t^Q is $\mathcal{F}_t$-measurable and $\mathbb{P}(\Delta U_t^Q = \Delta X_t|\mathcal{F}_t) = 1-\mathbb{P}(\Delta U_t^Q = \Delta Y_t|\mathcal{F}_t) = Q_t$, where $\Delta Z_t := Z_t - \lim_{s\uparrow t} Z_s$ stands for the jump at time t of a process $Z = (Z_t)_{t\geq 0}$. We denote by $T_Q = \inf\{t > 0 : U_t^Q < 0\}$ the ruin time of a control Q and by $\varphi_Q(x) = \mathbb{P}(T_Q = \infty)$ the corresponding survival probability. Given the starting point $x \geq 0$, the drifts $c_X, c_Y > 0$ and the Lévy measure Π (with marginals Π_X and Π_Y), the optimal underwriting problem consists of finding an optimal control/strategy Q^* such that the survival probability is maximised and to determine the corresponding maximal survival probability $\varphi_*(x)$,

$$\varphi_*(x) = \sup_Q \varphi_Q(x) = \varphi_{Q^*}(x).$$

The above optimal control problem is heavily motivated by the optimal new business problem of Hipp and Taksar [7]. Their problem corresponds to the case where $X = L_1$ and $Y = L_1 + L_2$ with L_1 and L_2 two independent spectrally negative Lévy processes of bounded variation (with finite Lévy measures). This matches the situation where an insurance company has its own existing business/portfolio with surplus process L_1 and can adaptively adjust the proportion of new business, represented by the surplus process L_2, it wants to take on. Another interpretation of this model is that the company can spend some capital towards the prevention of claims corresponding to L_2, see [4]. Note that the pair $(L_1, L_1 + L_2)$ is a bivariate Lévy process so the optimal new business problem of Hipp and Taksar [7] is contained in our setting.

It is intuitively clear from the problem statement that an optimal strategy should be Markovian, i.e. how to choose Q_t, the control at time t, should only depend on $U_{t-}^Q := \lim_{s\uparrow t} U_s^Q$, the state of the controlled process just prior to t. Further, as observed in [7], the control Q appears linearly in the associated Hamilton-Jacobi-Bellman equation (see also Equation (14) below) which means that an optimal control should be of bang-bang type, i.e. it takes values only in $\{0,1\}$. So essentially the optimal control problem boils down to an optimal switching problem with two modes X and Y and one has to decide when to switch from one mode to the other depending on the state of the controlled process. This implies further that the precise dependence structure between X and Y plays no role in the solution. One rather simple Markovian bang-bang strategy is the one where $Q_t = 1$ (mode X) if $U_{t-}^Q \leq b$ and $Q_t = 0$ (mode Y) if $U_{t-}^Q > b$ for some $b \in [0,\infty]$. Such a 1-switch-level strategy $Q^b = \{Q_t^b\}_{t\geq 0}$ at level b can be rigorously defined as follows. We let $U^b = \{U_t^b\}_{t\geq 0}$ be the process defined by

$$\mathrm{d}U_t^b = \mathbf{1}_{\{U_{t-}^b \leq b\}}\mathrm{d}X_t + \mathbf{1}_{\{U_{t-}^b > b\}}\mathrm{d}Y_t, \quad t > 0, \tag{5}$$

with $U_0^b = x$. Since X and Y are spectrally negative Lévy processes of bounded variation the point b is irregular for $(-\infty, b)$ (see e.g. p.155-158 in [10]), i.e. X and Y do not immediately go below b when hitting the level b. This ensures that for each $\omega \in \Omega$ there exists a unique solution $t \mapsto U_t^b(\omega)$ to the stochastic differential equation (5) given the sample paths $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$. This solution can be constructed piece-by-piece in a similar way as on p.30 in [9] which deals with the specific case where $Y_t = X_t - \delta t$ with $\delta > 0$, see also Section 1 in [15]. We can now define the threshold strategy Q^b by $Q_t^b = \mathbf{1}_{\{U_{t-}^b \leq b\}}$ for $t > 0$ and $Q_0^b = \mathbf{1}_{\{x \leq b\}}$. Note that U^b and thus Q^b is $\{\mathcal{F}_t\}$-adapted and that Q^b has càglàd sample paths, so Q^b is a control. One can easily show that the corresponding controlled process U^{Q^b} is equal to U^b, i.e. $U_t^{Q^b}(\omega) = U_t^b(\omega)$ for all $t \geq 0$ and $\omega \in \Omega$. Indeed, since $\int_{u=0}^1 \widetilde{N}(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}u) = N(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y)$ and $\{u \leq Q_s^b\} = \{U_{s-}^b \leq b\}$ for any $u \in (0, 1]$, we get via (3)-(5),

$$
\begin{aligned}
U_t^{Q^b} =& x + \int_0^t (Q_s^b c_X + (1 - Q_s^b) c_Y) \mathrm{d}s \\
& - \int_0^t \int_{[0,\infty)^2} \int_0^1 \left(x \mathbf{1}_{\{u \leq Q_s^b\}} + y \mathbf{1}_{\{u > Q_s^b\}} \right) \widetilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y, \mathrm{d}u) \\
=& x + \int_0^t \mathbf{1}_{\{U_{s-}^b \leq b\}} \mathrm{d} \left(c_X s - \int_0^s \int_{[0,\infty)^2} \int_0^1 x \widetilde{N}(\mathrm{d}r, \mathrm{d}x, \mathrm{d}y, \mathrm{d}u) \right) \\
& + \int_0^t \mathbf{1}_{\{U_{s-}^b > b\}} \mathrm{d} \left(c_Y s - \int_0^s \int_{[0,\infty)^2} \int_0^1 y \widetilde{N}(\mathrm{d}r, \mathrm{d}x, \mathrm{d}y, \mathrm{d}u) \right) \\
=& x + \int_0^t \mathbf{1}_{\{U_{s-}^b \leq b\}} \mathrm{d}X_s + \int_0^t \mathbf{1}_{\{U_{s-}^b > b\}} \mathrm{d}Y_s \\
=& U_t^b.
\end{aligned}
$$

For clarity we will sometimes denote Q^b by $Q^b(X, Y)$. By reversing the roles of X and Y we can define the (reversed) 1-switch-level strategy $Q^b(Y, X)$ for any $b \in [0, \infty]$. The main result of the paper is to show that a 1-switch-level strategy Q^b for some $b \in [0, \infty]$ is optimal (for any starting point $x \geq 0$) under certain easy-to-check conditions on the Lévy measures Π_X and Π_Y. In the next section after going through some preliminaries we will provide the precise statement and explain how this result relates to the existing literature. In particular we highlight that the seemingly easy case where $\mathbb{E}[X_1] \leq 0$ and $\mathbb{E}[Y_1] \leq 0$ is actually non-trivial. As part of the proof we will also in Section 2 establish an apparently new monotonicity property of renewal equations, which we believe is of independent interest. The proof of the main result itself will appear in Section 3 whereas some examples are treated in Section 4.

2 Main result

Before we state the main result we need to briefly introduce some concepts that will appear in the main theorem. We call a function $k : (0, \infty) \to [0, \infty)$ log-convex if $\log k$ is convex on $(0, \infty)$. It is well-known that k being log-convex is equivalent to,

$$
k(x_1 + y_1) k(x_2 + y_2) \geq k(x_2 + y_1) k(x_1 + y_2), \quad 0 \leq x_1 \leq x_2,\ 0 \leq y_1 \leq y_2,\ x_1 + y_1 > 0, \tag{6}
$$

see e.g. the argumentation in the proof of Lemma 1 in [18]. Note also that if $k:(0,\infty)\to[0,\infty)$ is log-convex and $k(x)=0$ for some $x>0$, then $k\equiv 0$, i.e. $k(x)=0$ for all $x>0$.

For Z a spectrally negative Lévy process of bounded variation we denote by $\overline{\Pi}_Z(z):=\Pi_Z(z,\infty)$, $z>0$, the tail of the Lévy measure Π_Z. We denote its Laplace transform by $\widehat{\Pi}_Z(\lambda)=\int_0^\infty \mathrm{e}^{-\lambda z}\overline{\Pi}_Z(z)\mathrm{d}z$, $\lambda>0$, which is well-defined by (2). We set $\widehat{\Pi}_Z(0):=\int_0^\infty \overline{\Pi}_Z(z)\mathrm{d}z\in(0,\infty]$.

We say that a function $f:(0,\infty)\to\mathbb{R}$ is locally bounded if f is bounded on $[a,b]$ for any $0<a<b<\infty$ and is locally integrable if f is measurable and $\int_0^x|f(y)|\mathrm{d}y<\infty$ for all $0<x<\infty$. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be in $\mathcal{D}$ if $f(x)=0$ for $x<0$ and f has a locally integrable and locally bounded right-derivative f' on $(0,\infty)$ which serves as a density for f on $[0,\infty)$, i.e. $f(x)-f(0)=\int_0^x f'(y)\mathrm{d}y$ for all $x>0$. The operator $\mathcal{A}_Z$ acting on a function $f\in\mathcal{D}$ is defined by

$$\begin{aligned}\mathcal{A}_Z f(x) :=& c_Z f'(x)+\int_0^\infty \left(f(x-z)-f(x)\right)\Pi_Z(\mathrm{d}z)\\ =& c_Z f'(x)-\int_0^x f'(x-z)\overline{\Pi}_Z(z)\mathrm{d}z-f(0)\overline{\Pi}_Z(x),\quad x>0,\end{aligned}\tag{7}$$

where the last line follows by a Fubini argument which can be applied as $f\in\mathcal{D}$ and (2) holds. It is easy to verify,

$$\int_0^\infty \mathrm{e}^{-\lambda x}\mathcal{A}_Z f(x)\mathrm{d}x=\left(c_Z-\widehat{\Pi}_Z(\lambda)\right)\lambda\int_0^\infty \mathrm{e}^{-\lambda x}f(x)\mathrm{d}x-c_Z f(0),\tag{8}$$

for $\lambda>0$ such that $\int_0^\infty \mathrm{e}^{-\lambda x}|f(x)|\mathrm{d}x<\infty$.

The scale function $W_Z:\mathbb{R}\to(0,\infty)$ of Z is defined by $W_Z(x)=0$ for $x<0$ and on $[0,\infty)$ it is characterised as the continuous function whose Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\lambda x}W_Z(x)\mathrm{d}x=\frac{1}{\lambda\left(c_Z-\widehat{\Pi}_Z(\lambda)\right)},\quad \lambda>\Phi_Z(0),\tag{9}$$

where $\Phi_Z(0)=\inf\{\lambda\geq 0:\widehat{\Pi}_Z(\lambda)\leq c_Z\}$, see Theorem 8.1(i) in [10]. From (8.26) in [10] it follows that $W_Z\in\mathcal{D}$ with the right-derivative W_Z' being strictly positive and right-continuous. Hence, $\mathcal{A}_Z W_Z$ is right-continuous on $(0,\infty)$ as $\overline{\Pi}_Z$ is right-continuous. Actually, since $W_Z(0)=1/c_Z$ (see Lemma 8.6 in [10]) it follows via (8) and (9) that the Laplace transform of the right-continuous function $\mathcal{A}_Z W_Z$ is identically 0 and thus,

$$\mathcal{A}_Z W_Z(x)=0,\quad x>0.\tag{10}$$

We also remark that by (1) and the compensation/master formula for Poisson point processes (see Proposition XII.1.10 in [16]),

$$\mathbb{E}[Z_1]=c_Z-\mathbb{E}\left[\int_0^1\int_0^\infty zN_Z(\mathrm{d}t,\mathrm{d}z)\right]=c_Z-\int_0^1\int_0^\infty z\mathrm{d}t\Pi_Z(\mathrm{d}z)=c_Z-\widehat{\Pi}_Z(0),\tag{11}$$

where the last equality follows by a similar argument as how one can obtain the equality in (2).

Our main result is the following. Note that terms like increasing and positive are always meant in the weak sense.

Theorem 2.1. *Assume $\overline{\Pi}_X$ is log-convex on $(0,\infty)$ and $x \mapsto \frac{\overline{\Pi}_X(x)}{\overline{\Pi}_Y(x)} \in (0,\infty]$ is increasing on $(0,\infty)$. Then the following holds with $b^* := \inf\{b>0 : \mathcal{A}_Y W_X(b) > 0\} \in [0,\infty]$.*

(i) For all $x > b^*$, $\mathcal{A}_Y W_X(x) \geq 0$.

(ii) If $\mathbb{E}[X_1] > 0$ or $\mathbb{E}[Y_1] > 0$, then $Q^{b^*} = Q^{b^*}(X,Y)$, the 1-switch-level strategy at level b^*, is optimal and the maximal survival probability is, for any starting point $x \geq 0$,

$$\varphi_*(x) = \varphi_{Q^{b^*}}(x) = \begin{cases} \frac{\mathbb{E}[Y_1]\left(W_X(x) - \int_{b^*}^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z\right)}{\frac{c_Y}{c_X} + \int_0^{b^*} \mathcal{A}_Y W_X(z)\mathrm{d}z} & \text{if } 0 < b^* < \infty, \\ \mathbb{E}[Y_1] W_Y(x) & \text{if } b^* = 0, \\ \mathbb{E}[X_1] W_X(x) & \text{if } b^* = \infty. \end{cases}$$

(iii) If $\mathbb{E}[X_1] \leq 0$ and $\mathbb{E}[Y_1] \leq 0$, then $\varphi_*(x) = 0$ for all $x \geq 0$, so ruin happens with probability 1 no matter the control chosen.

Theorem 2.1 gives a very explicit solution to the optimal control problem under conditions which are fairly easy to check; note also that part (i) says that if $0 < b^* < \infty$, then it is the point of sign change of a function that changes sign exactly once and so b^* is relatively easy to find numerically. The findings of Hipp and Taksar on their optimal new business problem in [7] are of a very different nature than ours: they focused on showing that in general a Markovian bang-bang strategy is optimal and in the process could state explicitly what the optimal mode is when the controlled process is at level 0 (but not at any other level). The optimal new business problem where X and Y are spectrally positive (instead of negative) Lévy processes of bounded variation and with finite Lévy measures has been covered in [14], where it was shown that it is optimal to acquire all the new business all the time or to never obtain it. Under a mild additional condition one can reformulate the conditions on the two Lévy measures in Theorem 2.1 in terms of their so-called hazard rates. Assuming the Lévy measure Π_Z has a density denoted by π_Z, we denote $r_Z(x) = \frac{\pi_Z(x)}{\overline{\Pi}_Z(x)}$, $x > 0$, and call it the hazard rate of Π_Z. It is easy to see that $\overline{\Pi}_X$ being log-convex is equivalent to Π_X having a decreasing hazard rate and that if Π_X and Π_Y both have densities, then $\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_Y(x)}$ being increasing is equivalent to $r_X(x) \leq r_Y(x)$ for a.e. $x > 0$. Recalling that X and Y play completely symmetric roles in the statement of the optimal control problem, we can conclude by Theorem 2.1 that if the hazard rates of Π_X and Π_Y are decreasing and ordered, then an optimal strategy is either formed by $Q^b(X,Y)$ or $Q^b(Y,X)$ for some $b \in [0,\infty]$, i.e. it is formed by a 1-switch-level strategy (though this includes the degenerate cases $b = 0$ and $b = \infty$ in which there is no switching at all). Sufficient conditions on the Lévy measure for optimality of simple strategies in optimal control problems for spectrally negative Lévy processes have been established before. For instance, a closely related optimal control problem is the one considered in e.g. [11], which is essentially an optimal switching problem with X and Y spectrally negative Lévy processes satisfying $Y_t = X_t - \delta t$, $t \geq 0$, for a given $\delta > 0$ and where the objective is maximising the expected, exponentially discounted amount of time spent in mode Y instead of minimising the ruin probability. For that optimal control problem it was shown in [11] that Q^b is optimal for some $0 \leq b < \infty$ if $\overline{\Pi}_X = \overline{\Pi}_Y$ is completely monotone, a stronger condition than log-convexity. Theorem 2.1 is the first result of this kind that deals with two Lévy processes with different Lévy measures. The condition of a decreasing hazard rate has appeared before in optimal control problems for spectrally negative Lévy processes (the first time was in [13]) but it is interesting to observe that $\overline{\Pi}_X$ and $\overline{\Pi}_Y$ both having decreasing hazard rates is not enough for establishing optimality of a 1-switch-level strategy but one needs in addition the two hazard rates to be ordered. Indeed, we give an example in Section 4 where the two hazard rates are decreasing but no 1-switch-level strategy is optimal.

Ruin occurs almost surely for a spectrally negative Lévy process Z if $\mathbb{E}[Z_1] \leq 0$. Therefore, if $\mathbb{E}[X_1] \leq 0$ and $\mathbb{E}[Y_1] \leq 0$, then surely, no matter how we (adaptively) switch between the two processes X and Y, ruin should happen with probability 1? Well, the answer is no when $\mathbb{E}[X_1] = \mathbb{E}[Y_1] = 0$. Durrett, Kesten and Lawler [3] have produced examples where by switching just deterministically between two mean-zero/oscillating random walks, one can survive with strictly positive probability. One can, in the obvious way, adjust the two-mode inhomogeneous random walk of Example 1 in [3] to our setting of two mean-zero spectrally negative Lévy processes with finite Lévy measures where one switches between the two depending on the number of jumps that have occurred and then mimic the arguments in Section 2 of [3] in order to show that the resulting process drifts to $+\infty$ a.s. and thus avoids ruin with strictly positive probability at any positive starting point. Part (iii) of Theorem 2.1 shows that, under the conditions of the theorem, this counterintuitive behaviour cannot happen. This complements Theorem 1 in [3] where such a result is established for random walks with finite variances.

As a secondary result to our main theorem we record the following proposition which gives a sufficient condition for when it is optimal to always be in the same mode. Its proof will be postponed to Section 3.

Proposition 2.2. *Assume* $\frac{\overline{\Pi}_X(x)}{c_X} \geq \frac{\overline{\Pi}_Y(x)}{c_Y}$ *for all* $x > 0$. *Then the strategy* Q^0 *(always mode* Y*) is optimal and the corresponding maximal survival probability is given by, for* $x \geq 0$,

$$\varphi_*(x) = \varphi_{Q^0}(x) = \begin{cases} \mathbb{E}[Y_1] W_Y(x) & \text{if } \mathbb{E}[Y_1] > 0, \\ 0 & \text{if } \mathbb{E}[Y_1] \leq 0. \end{cases}$$

The key tool for proving part (i) of Theorem 2.1 is the following monotonicity property for renewal equations with log-convex kernels, which we believe has applications beyond the optimal problem considered in this paper. This result gives conditions under which the forcing function f in (12) below having one sign-change implies that the solution u has at most one sign-change, or to be more precise, under which $\int_0^x f(y)\mathrm{d}y$ being quasi-convex implies the quasi-convexity of $\int_0^x u(y)\mathrm{d}y$. We have not been able to find this result in the literature; the closest result we could find is Theorem 3.3 in [6] which provides a monotonicity property for renewal equations with somewhat similar conditions.

Lemma 2.3. *Assume $k:(0,\infty)\to(0,\infty)$ is locally integrable and log-convex. Assume $f:(0,\infty)\to\mathbb{R}$ is bounded on sets of the form $(0,K]$, $K>0$. Assume in addition $x\mapsto\frac{f(x)}{k(x)}$ is increasing on $(0,\infty)$. Let $u:(0,\infty)\to\mathbb{R}$ be the unique locally integrable solution to the renewal equation,*

$$u(x)=f(x)+\int_0^x k(x-y)u(y)\mathrm{d}y,\quad x>0. \tag{12}$$

Then u has the following property: if $u(x_0)\geq 0$ for some $x_0>0$, then $u(x)\geq 0$ for all $x>x_0$.

Proof. Since k is locally integrable and f is bounded on sets of the form $(0,K]$ for $K>0$, the renewal equation (12) has a unique locally integrable solution which is given by

$$u(x)=f(x)+\int_0^x f(x-y)r(y)\mathrm{d}y,\quad x>0, \tag{13}$$

where r is the a.e. unique locally integrable solution to the renewal equation $r(x)=k(x)+\int_0^x k(x-y)r(y)\mathrm{d}y$, $x>0$, which is moreover (a.e.) positive since k is positive, see Theorems 2.3.1 and 2.3.5 in [5]. Further, $x\mapsto u(x)-f(x)=\int_0^x f(x-y)r(y)\mathrm{d}y$ is continuous and bounded on sets of the form $(0,K]$, $K>0$, see Corollary 2.2.3 in [5].

Since $\frac{f(x)}{k(x)}$ is increasing we must have that (i) there exists $\epsilon>0$ such that $f(x)\geq 0$ on $(0,\epsilon)$ or (ii) there exists $\epsilon>0$ such that $f(x)<0$ on $(0,\epsilon)$. In case (i) we must have that f is positive on $(0,\infty)$ since $\frac{f(x)}{k(x)}$ is increasing and thus by (13) u is positive. So the lemma is proved in case (i). Assume case (ii) and let $x_0:=\inf\{x>0:u(x)\geq 0\}$. Without loss of generality we can assume $x_0<\infty$. From (13) we see that $x_0>0$. Since $\frac{f(x)}{k(x)}$ is increasing and k is continuous (since it is (log-)convex), it follows that $f(x)=\frac{f(x)}{k(x)}k(x)$ has a right- and left-limit for any $x>0$ and then so does u since $u-f$ is continuous. Denote $u^+(x)=\lim_{y\downarrow x}u(y)$ and $f^+(x)=\lim_{y\downarrow x}f(y)$, $x>0$. Since $\frac{f(x)}{k(x)}$ is increasing and k is continuous, we must have $\lim_{y\uparrow x}f(y)\leq f(x)\leq f^+(x)$ for any $x>0$. Then, because $u-f$ is continuous, $\lim_{y\uparrow x}u(y)\leq u(x)\leq u^+(x)$ for any $x>0$. Therefore, $u^+(x_0)\geq 0$. Further, note that u^+ satisfies (12) with f replaced by f^+ and that $u-f=u^+-f^+$. So for $x>x_0$,

$$\begin{aligned}u^+(x)=&u^+(x)+\frac{k(x)}{k(x_0)}u^+(x_0)-\frac{k(x)}{k(x_0)}u^+(x_0)\\=&\frac{k(x)}{k(x_0)}u^+(x_0)+f^+(x)-\frac{k(x)}{k(x_0)}f^+(x_0)+\int_0^{x_0}\left(k(x-y)-\frac{k(x)}{k(x_0)}k(x_0-y)\right)u(y)\mathrm{d}y\\&+\int_{x_0}^x k(x-y)u(y)\mathrm{d}y.\end{aligned}$$

We see that $\widetilde{u}(t) := u^+(x_0+t)$ satisfies the renewal equation, $\widetilde{u}(t) = h(t) + \int_0^t k(t-y)\widetilde{u}(y)\mathrm{d}y$, $t \geq 0$, where

$$h(t) := \frac{k(x_0+t)}{k(x_0)}u^+(x_0) + f^+(x_0+t) - \frac{k(x_0+t)}{k(x_0)}f^+(x_0) \\ + \int_0^{x_0} u(y)\left(k(x_0+t-y) - \frac{k(x_0+t)}{k(x_0)}k(x_0-y)\right)\mathrm{d}y.$$

Since $h(t) \geq 0$ for $t > 0$ due to the assumptions on k and f in combination with (6) and because $u^+(x_0) \geq 0$ and $u(y) \leq 0$ for $y \in (0, x_0)$ by definition of x_0 and since h is further bounded on sets of the form $(0, K]$ for $K > 0$, it follows that $\widetilde{u}(t) \geq 0$ for $t > 0$ by the general form of the solution of such a renewal equation, see the beginning of this proof. Hence the lemma is proved for u^+ instead of u. To finish the proof, assume there exists $x_1 > 0$ and $x_2 > x_1$ such that $u(x_1) \geq 0$ and $u(x_2) < 0$. Then $u^+(x_1) \geq u(x_1) \geq 0$ and $\lim_{y\uparrow x_2} u(y) \leq u(x_2) < 0$, which implies that there exists $x \in (x_1, x_2)$ such that $u^+(x) < 0$. But this contradicts the property we have just proved for u^+. $\square$

3 Proofs

Like is typical for results of this type, we prove part (ii) in Theorem 2.1 by first establishing a verification lemma (Lemma 3.1 below) that gives sufficient conditions for a given function to be un upper bound of φ_*. As a second step we derive an analytic representation for the value function (i.e. the survival probability) of the candidate optimal control (see Lemma 3.2 below), which we then use in the third and final step to show that the aforementioned value function satisfies the conditions of the verification lemma under the conditions of the theorem. A notable difference in our approach in comparison to the existing literature like e.g. [13] and [11] is the following. In those references the third step is executed by using the monotonicity property of the scale function that is implied by the condition imposed on the Lévy measure, see Theorem 1.2 and Lemma 4.2 in [13] and Lemmas 2 and 7 in [11]. In contrast we will pretty much not use the monotonicity property of W_X implied by the log-convexity of $\overline{\Pi}_X$ (except to apply Lemma 3.3(i) below in the special case where $\mathbb{E}[Y_1] \leq 0$), but instead we use this assumption on the Lévy measure (in combination with the second assumption in Theorem 2.1) directly to complete the third step of the proof. Part (iii) of Theorem 2.1 will be proved by using the verification lemma to obtain an arbitrarily small upper bound for φ_*.

Lemma 3.1. *Let $w : [0, \infty) \to \mathbb{R}$ be a function such that after extending w to the whole real line by setting $w(x) = 0$ for $x < 0$, we have (i) $w \in \mathcal{D}$ with $w(0) \geq 0$, (ii) $\liminf_{x\to\infty} w(x) \geq 1$ and (iii) $\mathcal{A}_X w(x) \leq 0$ and $\mathcal{A}_Y w(x) \leq 0$ for all $x > 0$. Then $w(x) \geq \varphi_*(x)$ for all $x \geq 0$.*

Proof. Extend w to the whole real line by setting $w(x) = 0$ for $x < 0$. Let Q be an arbitrary control and denote $\Delta U_t^Q := U_t^Q - U_{t-}^Q$ and $U_t^{Q,c} := U_t^Q - \sum_{0<s\leq t} \Delta U_s^Q$. Fix $n \geq 1$ and let $T_Q^n = \inf\{t > 0 : U_t^Q \notin [0, n]\}$. Fix $\epsilon > 0$ and let $w_\epsilon : \mathbb{R} \to \mathbb{R}$ be given by $w_\epsilon(y) = w(y + \epsilon)$,

$y \in \mathbb{R}$. Further, fix $x \geq 0$. Then, since by (i) w_ϵ is on $[0,n]$ absolutely continuous with a bounded density, we have by an application of the change of variables formula (see e.g. Proposition 0.4.6 in [16][1]) and (4),

$$\begin{aligned}
w_\epsilon(U^Q_{t\wedge T^n_Q}) =& w_\epsilon(x) + \int_0^{t\wedge T^n_Q} w'_\epsilon(U^Q_{s-})\mathrm{d}U^{Q,c}_s + \sum_{0<s\leq t\wedge T^n_Q}\Big(w_\epsilon(U^Q_{s-}+\Delta U^Q_s) - w_\epsilon(U^Q_{s-})\Big)\\
=& w_\epsilon(x) + \int_0^{t\wedge T^n_Q} w'_\epsilon(U^Q_{s-})(Q_s c_X + (1-Q_s)c_Y)\mathrm{d}s\\
&+ \int_0^{t\wedge T^n_Q}\int_{[0,\infty)^2}\int_0^1 \Big(w_\epsilon\Big(U^Q_{s-} - x\mathbf{1}_{\{u\leq Q_s\}} - y\mathbf{1}_{\{u>Q_s\}}\Big) - w_\epsilon(U^Q_{s-})\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y,\mathrm{d}u)\\
=& w_\epsilon(x) + \int_0^{t\wedge T^n_Q}(Q_s\mathcal{A}_X + (1-Q_s)\mathcal{A}_Y)w_\epsilon(U^Q_{s-})\mathrm{d}s + M_t,
\end{aligned} \tag{14}$$

where M_t is given by

$$\begin{aligned}
M_t =& \int_0^{t\wedge T^n_Q}\int_{[0,\infty)^2}\int_0^1 \Big(w_\epsilon\Big(U^Q_{s-} - x\mathbf{1}_{\{u\leq Q_s\}} - y\mathbf{1}_{\{u>Q_s\}}\Big) - w_\epsilon(U^Q_{s-})\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y,\mathrm{d}u)\\
&- \int_0^{t\wedge T^n_Q}\int_{[0,\infty)^2}\int_0^1 \Big(w_\epsilon\Big(U^Q_{s-} - x\mathbf{1}_{\{u\leq Q_s\}} - y\mathbf{1}_{\{u>Q_s\}}\Big) - w_\epsilon(U^Q_{s-})\Big)\mathrm{d}s\Pi(\mathrm{d}x,\mathrm{d}y)\mathrm{d}u
\end{aligned}$$

and where for the last step we used the identity

$$\begin{aligned}
&\int_0^{t\wedge T^n_Q}\int_{[0,\infty)^2}\int_0^1 \Big(w_\epsilon\Big(U^Q_{s-} - x\mathbf{1}_{\{u\leq Q_s\}} - y\mathbf{1}_{\{u>Q_s\}}\Big) - w_\epsilon(U^Q_{s-})\Big)\mathrm{d}s\Pi(\mathrm{d}x,\mathrm{d}y)\mathrm{d}u\\
&\qquad = \int_0^{t\wedge T^n_Q}\int_0^\infty Q_s\Big(w_\epsilon(U^Q_{s-} - x) - w_\epsilon(U^Q_{s-})\Big)\Pi_X(\mathrm{d}x)\mathrm{d}s\\
&\qquad\quad + \int_0^{t\wedge T^n_Q}\int_0^\infty (1-Q_s)\Big(w_\epsilon(U^Q_{s-} - y) - w_\epsilon(U^Q_{s-})\Big)\Pi_Y(\mathrm{d}y)\mathrm{d}s,
\end{aligned}$$

which holds by evaluating the integral with respect to u and recalling that Π_X and Π_Y are the marginals of Π. As a process in s the integrand of the first integral above is $\{\mathcal{F}_t\}_{t\geq 0}$ adapted and has càglàd sample paths. Further, since by assumption (i) w_ϵ is bounded on $(-\infty, n]$ and w'_ϵ is bounded on $[-\epsilon/2, n]$ we can, in combination with Π_X and Π_Y satisfying (2), deduce that the expected value of the two integrals on the right hand side (and thus the integral on the left hand side) of the last equation is finite. Hence we can use the compensation/master formula of Poisson point processes (see Proposition XII.1.10 in [16]) to conclude that $\mathbb{E}[M_t] = 0$. Hence by taking expectations on both sides of (14) and using

[1] In this reference w'_ϵ is assumed to be bounded and continuous but from this case one can extend straightforwardly the given formula to the the case where w'_ϵ is merely bounded and measurable. Further note that we only need w'_ϵ to be bounded on $[0,n]$ since in our application of this formula we stop at time T^n_Q.

assumption (iii), we get $w(x+\epsilon) \geq \mathbb{E}\left[w(U^Q_{t\wedge T^n_Q}+\epsilon)\right]$. Further, since w is right-continuous on $\mathbb{R}$ we have by taking $\epsilon \downarrow 0$ and invoking the dominated convergence theorem,

$$w(x) \geq \mathbb{E}\left[w(U^Q_{t\wedge T^n_Q})\right]. \tag{15}$$

Now the process U^Q will eventually leave the interval $[0,n]$. Indeed, fix $h > n/(c_X \wedge c_Y)$. Then for $k = 0,1,2,\ldots$,

$$\begin{aligned}\left\{U^Q_{(k+1)h} - U^Q_{kh} > n\right\} &\supset \left\{\sum_{kh<s\leq(k+1)h} \Delta U^Q_s < (c_X\wedge c_Y)h - n\right\} \\ &\supset \left\{\int_{kh}^{(k+1)h}\int_{[0,\infty)^2}(x+y)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y) < (c_X\wedge c_Y)h - n\right\}.\end{aligned}$$

Denote the event on the right hand side by A_k. By properties of the Poisson point process N, the events $A_0, A_1, A_2, \ldots$ are independent and $\mathbb{P}(A_k) = \mathbb{P}(A_l) > 0$ for all $k,l \geq 0$ so that $\sum_{k\geq 0}\mathbb{P}(A_k) = \infty$. Hence by the second Borel-Cantelli Lemma (see e.g. Theorem 4.4 in [1]),

$$\mathbb{P}(U^Q_t \notin [0,n] \text{ for some } t > 0) \geq \mathbb{P}\left(\limsup_{k\to\infty}\left\{U^Q_{(k+1)h} - U^Q_{kh} > n\right\}\right) \geq \mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 1.$$

Hence U^Q will eventually leave the interval $[0,n]$, i.e. $T^n_Q < \infty$ almost surely and consequently, with $T^n := \inf\{t > 0 : U^Q_t > n\}$, we have $\mathbb{P}(T_Q = \infty) = \mathbb{P}(\cap_{n=1}^\infty\{T_Q > T_n\})$ since $T^n \to \infty$ a.s. as $n \to \infty$. Therefore, by taking limits as $t \to \infty$ in (15), we get via the dominated convergence theorem in combination with $w(x) = 0$ for $x < 0$, $w(0) \geq 0$ and U^Q having no upward jumps,

$$w(x) \geq \mathbb{E}\left[w(U^Q_{T^n_Q})\right] = \mathbb{E}\left[w(n)\mathbf{1}_{\{T_Q > T^n\}}\right] + \mathbb{E}\left[w(U^Q_{T^Q})\mathbf{1}_{\{T_Q < T^n\}}\right] \geq w(n)\mathbb{P}(T_Q > T^n).$$

Finally, by using assumption (ii), Fatou's lemma and the equality $\mathbb{P}(T_Q = \infty) = \mathbb{P}(\cap_{n=1}^\infty\{T_Q > T_n\})$,

$$w(x) \geq \left(\liminf_{n\to\infty} w(n)\right)\left(\liminf_{n\to\infty}\mathbb{P}(T_Q > T^n)\right) \geq \mathbb{P}(T_Q = \infty) = \varphi_Q(x).$$

Since the control Q and $x \geq 0$ were chosen arbitrarily, the conclusion of the lemma follows. □

Lemma 3.2. *For any $x \geq 0$,*

$$\varphi_{Q^b}(x) = \begin{cases} \dfrac{\mathbb{E}[Y_1]\left(W_X(x) - \int_b^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z\right)}{\frac{c_Y}{c_X} + \int_0^b \mathcal{A}_Y W_X(z)\mathrm{d}z} & \textit{if } 0 < b < \infty \textit{ and } \mathbb{E}[Y_1] > 0, \\ \mathbb{E}[Y_1]W_Y(x) & \textit{if } b = 0 \textit{ and } \mathbb{E}[Y_1] > 0, \\ \mathbb{E}[X_1]W_X(x) & \textit{if } b = \infty \textit{ and } \mathbb{E}[X_1] > 0. \end{cases}$$

Proof. For a strong Markov process $Z = (Z_t)_{t\geq 0}$ we use the notation $\tau_0^-(Z) = \inf\{t > 0 : Z_t < 0\}$ and $\tau_a^+ = \inf\{t > 0 : Z_t > a\}$. In what follows we will frequently use the following identity when Z is a spectrally negative Lévy process: for $a > 0$,

$$\mathbb{P}(\tau_a^+(Z) < \tau_0^-(Z)|Z_0 = x) = \frac{W_Z(x)}{W_Z(a)}, \quad x \leq a, \tag{16}$$

see e.g. (8.11) in [10] for a proof of this statement. Recall from the end of Section 1 that $U^{Q^b} = U^b$ where U^b is defined by (5) and so we need to compute $\mathbb{P}(\tau_0^-(U^b) = \infty|U_0^b = x)$. If $b = 0$, then $U^b = Y$ and the second case of the lemma follows then from the well-known identity for the ruin probability of a spectrally negative Lévy process, see e.g. (8.10) and p.231 in [10]. Similarly, the third case of the lemma follows since $U^b = X$ when $b = \infty$. Now assume $0 \leq b < \infty$. Since the bivariate process (X, Y) has stationary and independent increments, it follows that U^b is a strong Markov process. Fix $a > b$ and denote $p(x, b, a) := \mathbb{P}(\tau_a^+(U^b) < \tau_0^-(U^b)|U_0^b = x)$. If $x \leq b$, then $(U_t)_{0\leq t\leq \tau_b^+(U)} = (X_t)_{0\leq t\leq \tau_b^+(X)}$ and so we have by the strong Markov property of U^b and (16), for $x \leq b$,

$$p(x, b, a) = \mathbb{P}(\tau_b^+(X) < \tau_0^-(X)|X_0 = x)p(b, b, a) = \frac{W_X(x)}{W_X(b)}p(b, b, a). \tag{17}$$

Similarly, since for $x \geq b$, $(U_t)_{0\leq t\leq \tau_b^-(U^b)} = (Y_t)_{0\leq t\leq \tau_b^-(Y)}$, we have by the strong Markov property of U^b, for $b \leq x \leq a$,

$$\begin{aligned} p(x, b, a) =&\mathbb{P}(\tau_a^+(Y) < \tau_b^-(Y)|Y_0 = x) + \mathbb{E}\left[\mathbf{1}_{\{\tau_b^-(Y)<\tau_a^+(Y)\}}p(Y_{\tau_b^-(Y)}, b, a)\middle| Y_0 = x\right] \\ =&\frac{W_Y(x-b)}{W_Y(a-b)} + \frac{p(b,b,a)}{W_X(b)}\mathbb{E}\left[\mathbf{1}_{\{\tau_b^-(Y)<\tau_a^+(Y)\}}W_X(Y_{\tau_b^-(Y)})\middle| Y_0 = x\right] \\ =&\frac{W_Y(x-b)}{W_Y(a-b)} + \frac{p(b,b,a)}{W_X(b)}\Bigg\{W_X(x) - \int_b^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z \\ &- \frac{W_Y(x-b)}{W_Y(a-b)}\left(W_X(a) - \int_b^a W_Y(a-z)\mathcal{A}_Y W_X(z)\mathrm{d}z\right)\Bigg\}, \end{aligned} \tag{18}$$

where we used (17) in the second equality and Corollary 3 in [12] with $\widetilde{f} = W_X$ in the last one; note that the required smoothness condition in this reference is satisfied since $W_X \in \mathcal{D}$. Since $W_Y(0) = 1/c_Y > 0$, we deduce from (18) with $x = b$,

$$p(b, b, a) = \frac{W_X(b)}{W_X(a) - \int_b^a W_Y(a-z)\mathcal{A}_Y W_X(z)\mathrm{d}z}.$$

Plugging the last equality into (17) and (18) yields, for all $x \leq a$,

$$p(x, b, a) = \frac{W_X(x) - \int_b^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z}{W_X(a) - \int_b^a W_Y(a-z)\mathcal{A}_Y W_X(z)\mathrm{d}z}.$$

In order to finish the proof we need to take limits as $a \to \infty$. To this end note, since the last equality holds for $b = 0$ and $p(0, 0, a) = \frac{W_Y(0)}{W_Y(a)}$ by the fact $U^0 = Y$ and (16), we have

$$W_X(a) - \int_0^a W_Y(a-z)\mathcal{A}_Y W_X(z)\mathrm{d}z = \frac{W_Y(a)W_X(0)}{W_Y(0)} = W_Y(a)\frac{c_Y}{c_X}, \quad a \geq 0. \tag{19}$$

Consequently, if $\mathbb{E}[Y_1] > 0$, for $0 \leq x \leq b$,

$$\begin{aligned}\varphi_b(x) = \lim_{a\to\infty} p(x, b, a) =& \frac{W_X(x) - \int_b^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z}{\lim_{a\to\infty}\left(W_Y(a)\frac{c_Y}{c_X} + \int_0^b W_Y(a-z)\mathcal{A}_Y W_X(z)\mathrm{d}z\right)} \\ =& \mathbb{E}[Y_1]\frac{W_X(x) - \int_b^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z}{\frac{c_Y}{c_X} + \int_0^b \mathcal{A}_Y W_X(z)\mathrm{d}z},\end{aligned}$$

where we used that $\lim_{x\to\infty} W_Y(x) = 1/\mathbb{E}[Y_1]$ if $\mathbb{E}[Y_1] > 0$ (this can be seen from the second case of the lemma) in combination with the dominated convergence theorem in the last line. □

We need the following lemma to deal with the case where $\mathbb{E}[X_1] \leq 0$ or $\mathbb{E}[Y_1] \leq 0$.

Lemma 3.3. *(i) If W_X is concave on $(0, \infty)$, then $\limsup_{x\to\infty} \frac{\mathcal{A}_Y W_X(x)}{W_X'(x)} \leq \mathbb{E}[Y_1]$.*

(ii) If $\mathbb{E}[X_1] \leq 0$ and $\mathbb{E}[Y_1] > -\infty$, then $\lim_{x\to\infty} \frac{\int_0^x \mathcal{A}_Y W_X(z)\mathrm{d}z}{W_X(x)} = c_Y - \widehat{\Pi}_Y(\Phi_X(0)) \geq \mathbb{E}[Y_1]$.

Proof. (i). If W_X is concave on $(0, \infty)$, then W_X' is decreasing and so by (7), for $x > 0$,

$$\begin{aligned}\frac{\mathcal{A}_Y W_X(x)}{W_X'(x)} =& c_Y\frac{W_X'(x)}{W_X'(x)} - \int_0^x \frac{W_X'(x-z)}{W_X'(x)}\overline{\Pi}_Y(z)\mathrm{d}z - W_X(0)\frac{\overline{\Pi}_Y(x)}{W_X'(x)} \\ \leq& c_Y - \int_0^x \overline{\Pi}_Y(z)\mathrm{d}z.\end{aligned}$$

Hence by (11), $\limsup_{x\to\infty} \frac{\mathcal{A}_Y W_X(x)}{W_X'(x)} \leq c_Y - \int_0^\infty \overline{\Pi}_Y(z)\mathrm{d}z = \mathbb{E}[Y_1]$.

(ii). We first prove that $\frac{W_X(x-z)}{W_X(x)}$ increases monotonically to $\mathrm{e}^{-\Phi_X(0)z}$ as $0 < x \to \infty$ for a.e. $z > 0$. By (16) we have $\mathbb{P}(\tau_x^+(X) < \tau_0^-(X)|X_0 = x - z) = \frac{W_X(x-z)}{W_X(x)}$ for $x > z \geq 0$. Hence $x \mapsto \frac{W_X(x-z)}{W_X(x)}$ is increasing in $x > 0$ for any $z \geq 0$ and thus the limit $L(z) := \lim_{x\to\infty} \frac{W_X(x-z)}{W_X(x)} \in [0, 1]$ exists. If $\mathbb{E}[X_1] = 0$, then $\Phi_X(0) = 0$ and X is recurrent and thus oscillating, i.e. $\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty$, see e.g. Theorem 36.7 in [17]. Hence $L(z) = \lim_{x\to\infty} \mathbb{P}(\tau_x^+(X) < \tau_0^-(X)|X_0 = x - z) = 1 = \mathrm{e}^{-\Phi_X(0)z}$. If $\mathbb{E}[X_1] < 0$, then $\frac{W_X(x)}{\mathrm{e}^{\Phi_X(0)x}}$ is a bounded function on $(0, \infty)$ (see e.g. p.236 in [10]) and $\Phi_X(0) > 0$, which implies $L(z) = \mathrm{e}^{-\Phi_X(0)z}$. So the claim at the beginning is proved. Now let

$$F_{X,Y}(x) = \frac{c_Y}{c_X}\overline{\Pi}_X(x) - \overline{\Pi}_Y(x), \quad x > 0. \tag{20}$$

We have by (7) and (10), for $x>0$,

$$\int_0^x \mathcal{A}_Y W_X(z)\mathrm{d}z = \int_0^x \mathcal{A}_Y W_X(z)\mathrm{d}z - \frac{c_Y}{c_X}\int_0^x \mathcal{A}_X W_X(z)\mathrm{d}z = \int_0^x W_X(x-z)F_{X,Y}(z)\mathrm{d}z.$$

Hence by the monotone convergence theorem and the claim at the beginning, we have, provided $\mathbb{E}[Y_1]>-\infty$,

$$\begin{aligned}\lim_{x\to\infty}\frac{\int_0^x \mathcal{A}_Y W_X(z)\mathrm{d}z}{W_X(x)} =& \int_0^\infty \mathrm{e}^{-\Phi(0)z}F_{X,Y}(z)\mathrm{d}z\\ =&\frac{c_Y}{c_X}\widehat{\Pi}_X(\Phi_X(0)) - \widehat{\Pi}_Y(\Phi_X(0))\\ =&c_Y - \widehat{\Pi}_Y(\Phi_X(0)),\end{aligned}$$

where the last equality follows by the assumption $\mathbb{E}[X_1]\le 0$ in combination with (11) and the definition of $\Phi_X(0)$. The required inequality is due to (11). □

Proof of Theorem 2.1. (i). By (7), (10) and the fact $W_X(0)=1/c_X$,

$$\begin{aligned}c_X\mathcal{A}_Y W_X(x) =&c_X\mathcal{A}_Y W_X(x) - c_Y\mathcal{A}_X W_X(x)\\ =&\int_0^x W_X'(x-z)\left(c_Y\overline{\Pi}_X(z)-c_X\overline{\Pi}_Y(z)\right)\mathrm{d}z + W_X(0)\left(c_Y\overline{\Pi}_X(x)-c_X\overline{\Pi}_Y(x)\right)\\ =&F_{X,Y}(x) + \int_0^x F_{X,Y}(x-z)c_X W_X'(z)\mathrm{d}z,\end{aligned} \tag{21}$$

where $F_{X,Y}$ is defined in (20). We further have that $c_X\mathcal{A}_Y W_X(x)$ satisfies the renewal equation,

$$c_X\mathcal{A}_Y W_X(x) = F_{X,Y}(x) + \int_0^x \frac{\overline{\Pi}_X(x-z)}{c_X}c_X\mathcal{A}_Y W_X(z)\mathrm{d}z, \quad x>0. \tag{22}$$

Indeed by taking Laplace transforms on both sides of (22) and using (8) and (9) one sees that (22) holds for a.e. $x>0$. The right-continuity of $\mathcal{A}_Y W_X(x)$ then yields (22) for all $x>0$. By assumption $\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_Y(x)}$ is increasing, which implies $\frac{F_{X,Y}(x)}{\overline{\Pi}_X(x)}$ is increasing. Also $\overline{\Pi}_X$ is log-convex by assumption which implies further that $\overline{\Pi}_X$ is strictly positive since $\overline{\Pi}_X\not\equiv 0$ because otherwise X does not have non-monotone sample paths. Since $F_{X,Y}$ is not in general bounded on sets of the form $(0,K]$, $K>0$, but merely locally bounded, Lemma 2.3 does not apply. However, given (22) the proof of Lemma 2.3 after the first two sentences goes through verbatim for $u(x):=c_X\mathcal{A}_Y W_X(x)$ once we replace (13) by (21) and note that the integral term in (21) is continuous on $(0,\infty)$ because $F_{X,Y}$ and W_X' are locally integrable and locally bounded on $(0,\infty)$. Hence the conclusion of Lemma 2.3 holds for $u(x)=c_X\mathcal{A}_Y W_X(x)$. This proves part (i) of the theorem.

(ii). Assume $\mathbb{E}[X_1]>0$ or $\mathbb{E}[Y_1]>0$. If $\mathbb{E}[Y_1]\le 0$ then $\mathbb{E}[X_1]>0$ which means that W_X is bounded on $(0,\infty)$ (see the third case of Lemma 3.2). Since moreover W_X' is log-convex as $\overline{\Pi}_X$ is log-convex (see Theorem 1.2 in [13]) it follows that W_X' is decreasing, i.e. W_X is concave. So, if $\mathbb{E}[Y_1] < 0$, then by Lemma 3.3(i), $\mathcal{A}_Y W_X(x) < 0$ for x sufficiently large, which implies by part (i) of the theorem that $\mathcal{A}_Y W_X(x) < 0$ for all $x > 0$. If $\mathbb{E}[Y_1] = 0$, then $Y - \delta = (Y_t - \delta t)_{t\geq 0}$ where $0 < \delta < c_Y$, is a spectrally negative Lévy process of bounded variation and by what we have just proved, $\mathcal{A}_{Y+\delta} W_X(x) < 0$ for all $x > 0$. Hence $\mathcal{A}_Y W_X(x) = \lim_{\delta\downarrow 0} \mathcal{A}_{Y+\delta} W_X(x) \leq 0$ for all $x > 0$. We can conclude that $b^* = \infty$ if $\mathbb{E}[Y_1] \leq 0$. If $\mathbb{E}[X_1] \leq 0$, then $\mathbb{E}[Y_1] > 0$ and by Lemma 3.3(ii), $\int_0^x \mathcal{A}_Y W_X(z)\mathrm{d}z > 0$ for some $x > 0$, which implies $b^* < \infty$. Denote for convenience $\varphi_b := \varphi_{Q^b}$. Given Lemma 3.2 and what we have just showed, we can now conclude that once we show that $w = \varphi_{b^*}$ satisfies the three conditions of Lemma 3.1, part (ii) of the theorem follows by Lemma 3.1. For a Lévy process $Z = (Z_t)_{t\geq 0}$, $\mathbb{E}[Z_1] > 0$ implies $\lim_{t\to\infty} Z_t = \infty$ a.s by the strong law of large numbers, see e.g. Theorem 36.5 in [17]. So $\lim_{x\to\infty} \varphi_{b^*}(x) = 1$ since $b^* < \infty$ implies $\mathbb{E}[Y_1] > 0$ and $b^* = \infty$ implies $\mathbb{E}[X_1] > 0$. Hence condition (ii) of Lemma 3.1 is satisfied. Since $W_X, W_Y \in \mathcal{D}$, condition (i) of Lemma 3.1 is satisfied if $b^* = 0$ or $b^* = \infty$. If $0 < b^* < \infty$, then by (19) we can write, for $x \geq b^*$,

$$\varphi_{b^*}(x) = \frac{\mathbb{E}[Y_1]}{\frac{c_Y}{c_X} + \int_0^{b^*} \mathcal{A}_Y W_X(z)\mathrm{d}z} \left(\frac{c_Y}{c_X} W_Y(x) + \int_0^{b^*} W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z \right). \tag{23}$$

Note that by (19) and since $W_Y(y) = 0$ for $y < 0$ we can show that (23) actually holds for any $x \geq 0$. So the right-derivative of φ_{b^*} on $(0,\infty)$ exists and is given by

$$\varphi_{b^*}'(x) = \begin{cases} \frac{\mathbb{E}[Y_1]}{\frac{c_Y}{c_X}+\int_0^{b^*} \mathcal{A}_Y W_X(z)\mathrm{d}z} W_X'(x) & \text{if } 0 < x < b^*, \\ \frac{\mathbb{E}[Y_1]}{\frac{c_Y}{c_X}+\int_0^{b^*} \mathcal{A}_Y W_X(z)\mathrm{d}z} \left(W_X'(b^*) - W_Y(0)\mathcal{A}_Y W_X(b^*)\right) & \text{if } x = b^*, \\ \frac{\mathbb{E}[Y_1]}{\frac{c_Y}{c_X}+\int_0^{b^*} \mathcal{A}_Y W_X(z)\mathrm{d}z} \left(\frac{c_Y}{c_X} W_Y'(x) + \int_0^{b^*} W_Y'(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z \right) & \text{if } x > b^*, \end{cases}$$

which is clearly locally bounded and forms a density of φ_{b^*} on $[0,\infty)$. Hence condition (ii) of Lemma 3.1 is also satisfied if $0 < b^* < \infty$.

It remains to show that condition (iii) of Lemma 3.1 is satisfied. Assume $b^* = 0$. Then $\mathcal{A}_Y W_X(x) \geq 0$ for all $x > 0$, which implies by (21) that for all $\epsilon > 0$ there exists $0 < x < \epsilon$ such that $F_{X,Y}(x) \geq 0$. Since $\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_Y(x)}$ is increasing by hypothesis, we therefore have that $F_{X,Y}(x) \geq 0$ for all $x > 0$. This implies $F_{Y,X}(x) \leq 0$ for all $x > 0$, which implies by reversing the roles of X and Y in (21) that $\mathcal{A}_X W_Y(x) \leq 0$ for all $x > 0$. Hence $\mathcal{A}_X \varphi_0(x) \leq 0$ for all $x > 0$ and since $\mathcal{A}_Y \varphi_0(x) = 0$ by (10) for $x > 0$, we conclude that condition (iii) of Lemma 3.1 is satisfied if $b^* = 0$. Further, for any $x > 0$, $\mathcal{A}_X \varphi_\infty(x) = 0$ by (10) and $\mathcal{A}_Y \varphi_\infty(x) \leq 0$ if $b^* = \infty$ by definition of b^*. Hence condition (iii) in Lemma 3.1 is also satisfied if $b^* = \infty$. Now assume the remaining case $0 < b^* < \infty$. For $0 < x < b^*$, we have $\mathcal{A}_X \varphi_{b^*}(x) = 0$ by (10) and $\mathcal{A}_Y \varphi_{b^*}(x) \leq 0$ by definition of b^*. For $x = b^*$, $\mathcal{A}_X \varphi_{b^*}(x) \leq 0$ by (10) and since $\mathcal{A}_Y W_X(b^*) \geq 0$, where we note that the latter holds by definition of b^* and the right-continuity of $\mathcal{A}_Y W_X$ on $(0,\infty)$. It is easy to check that for $x = b^*$, $\mathcal{A}_Y \varphi_{b^*}(x) = 0$. For $x > b^*$, we have $\mathcal{A}_Y \varphi_{b^*}(x) = 0$ by (23) (which recall holds for all $x \geq 0$) and (10); note that it is straightforward to show that one can take the operator $\mathcal{A}_Y$ inside the integral in (23). It remains to show that $\mathcal{A}_X\varphi_{b^*}(x)\leq 0$ for $x>b^*$ which is the key part of the proof in the case $0<b^*<\infty$. To this end, we can write, for $x>b^*$, with $K=\frac{\mathbb{E}[Y_1]}{\frac{c_Y}{c_X}+\int_0^{b^*}\mathcal{A}_Y W_X(z)\mathrm{d}z}>0$,

$$\begin{aligned}\frac{1}{K}\mathcal{A}_X\varphi_{b^*}(x)&=-\int_0^{x-b^*}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{b^*}^{x-y}W_Y(x-y-z)\mathcal{A}_Y W_X(z)\mathrm{d}z\right)\Xi_X(\mathrm{d}y)\\&=-\int_0^{x-b^*}\left(\int_{b^*}^{x-y}W_Y(x-y-\mathrm{d}z)\mathcal{A}_Y W_X(z)\right)\Xi_X(\mathrm{d}y)\\&=-\int_{b^*}^{x}W_Y(x-\mathrm{d}u)\left(\int_0^{u-b^*}\mathcal{A}_Y W_X(u-y)\Xi_X(\mathrm{d}y)\right)\\&=-\int_{b^*}^{x}W_Y(x-\mathrm{d}u)\left(F_{X,Y}(u)-\int_{u-b^*}^{u}\mathcal{A}_Y W_X(u-y)\Xi_X(\mathrm{d}y)\right)\\&=-\int_{b^*}^{x}W_Y(x-\mathrm{d}u)\left(F_{X,Y}(u)+\int_0^{b^*}\mathcal{A}_Y W_X(z)\overline{\Pi}_X(u-z)\mathrm{d}z\right),\end{aligned}$$

where for the first equality we used Lemma 3.2, (10), (7), $\int_{b^*}^x W_Y(x-z)\mathcal{A}_Y W_X(z)\mathrm{d}z=0$ for $x\leq b^*$ and the notation $\Xi_X(\mathrm{d}y)=c_X\delta_0(\mathrm{d}y)-\overline{\Pi}_X(y)\mathrm{d}y$ where $\delta_a(\mathrm{d}y)$ stands for the Dirac mass at a, in the second equality we differentiated the convolution (see e.g. Lemma 2.4 in [2]) and used the notation $W_Y(a-\mathrm{d}z):=W_Y'(a-z)\mathrm{d}z+W_Y(0)\delta_a(\mathrm{d}z)$, for the third equality we used Fubini and the change of variables $u=z+y$, in the fourth equality we used (22) and finally we used a change of variables for the last equality. Now define

$$h(x)=F_{X,Y}(x)+\int_0^{b^*}\mathcal{A}_Y W_X(z)\overline{\Pi}_X(x-z)\mathrm{d}z,\quad x\geq b^*.$$

We are done if we show $h(x)\geq 0$ for all $x\geq b^*$. By (22), the definition of b^* and the right-continuity of $\mathcal{A}_Y W_X$, $h(b^*)=c_X\mathcal{A}_Y W_X(b^*)\geq 0$. Therefore, for any $x\geq b^*$,

$$\begin{aligned}h(x)\geq h(x)-\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_X(b^*)}h(b^*)=\int_0^{b^*}\mathcal{A}_Y W_X(z)\left[\overline{\Pi}_X(x-z)-\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_X(b^*)}\overline{\Pi}_X(b^*-z)\right]\mathrm{d}z\\+\frac{\overline{\Pi}_X(x)}{\overline{\Pi}_X(b^*)}\overline{\Pi}_Y(b^*)-\overline{\Pi}_Y(x).\end{aligned}$$

By the assumptions of the theorem in combination with (6) and since $\mathcal{A}_Y W_X(z)\leq 0$ for $z\in(0,b^*)$ by definition of b^*, it follows that $h(x)\geq 0$ for all $x\geq b^*$.

(iii) We can assume without loss of generality $\mathbb{E}[X_1]=\mathbb{E}[Y_1]=0$ because otherwise one can increase c_X and/or c_Y in order to get to this case without lowering the maximal survival probability. We first show that $W_X(x)$ is concave for $x>0$ and increases to infinity as $x\to\infty$. Since $\mathbb{E}[X_1]=0$, we have $\limsup_{t\to\infty}X_t=-\liminf_{t\to\infty}X_t=\infty$ a.s. and thus by (16),

$$0=\lim_{x\to\infty}\mathbb{P}(\tau_x^+(X)<\tau_0^-(X)|X_0=1)=\lim_{x\to\infty}\frac{W_X(1)}{W_X(x)}.$$

Hence $\lim_{x\to\infty} W_X(x) = \infty$. Consider the spectrally negative Lévy process $X + \delta = (X_t + \delta t)_{t\geq 0}$ where $\delta > 0$. By (19),

$$W_X(x) - \delta \int_0^x W_{X+\delta}(x-z) W_X'(z) \mathrm{d}z = W_{X+\delta}(x) + \frac{\delta}{c_X} W_{X+\delta}(x), \quad x > 0,$$

from which we see that $W_{X+\delta}(x)$ increases monotonically to $W_X(x)$ as $\delta \downarrow 0$ for any $x > 0$. As argued in the beginning of the proof of part (ii), $W_{X+\delta}(x)$ is concave for $x > 0$ since $\mathbb{E}[X_1 + \delta] > 0$. Because the pointwise limit of a convergent sequence of concave functions is concave, it follows that W_X is concave on $(0, \infty)$. Now fix $\varepsilon > 0$ and let $w(x) = \varepsilon W_X(x)$. Then w satisfies condition (i) of Lemma 3.1 and, since $\lim_{x\to\infty} W_X(x) = \infty$, condition (ii) of Lemma 3.1 is also satisfied. Further, for any $x > 0$, $\mathcal{A}_X w(x) = 0$ by (10) and since W_X is still concave if $\mathbb{E}[X_1] = 0$ we can use the same arguments as in the beginning of the proof of part (ii) to show that $\mathcal{A}_Y w(x) \leq 0$ for any $x > 0$. So by Lemma 3.1, $\varepsilon W_X(x) = w(x) \geq \varphi_*(x)$ for all $x \geq 0$. Since $\varepsilon > 0$ was chosen arbitrarily, it follows that $\varphi_*(x) = 0$ for all $x \geq 0$. □

Proof of Proposition 2.2. By assumption $F_{X,Y}$ as defined in (20) is a positive function or, equivalently, $F_{Y,X}$ is a negative function. Then by reversing the roles of X and Y in (21) we deduce that $\mathcal{A}_X W_Y(x) \leq 0$ for all $x > 0$. By following the $b^* = 0$ case of the proof of Theorem 2.1(ii) it is then straightforward to show, without using the assumptions of Theorem 2.1, that $w(x) = \mathbb{E}[Y_1] W_Y(x)$ satisfies the three conditions of Lemma 3.1 if $\mathbb{E}[Y_1] > 0$. This implies Q^0 is an optimal control if $\mathbb{E}[Y_1] > 0$. Similarly, if $\mathbb{E}[Y_1] \leq 0$ we can easily show that, for any $\varepsilon > 0$, $w(x) = \varepsilon W_Y(x)$ satisfies the three conditions of Lemma 3.1; in particular $\lim_{x\to\infty} W_Y(x) = \infty$ follows by the same arguments as in the proof of Theorem 2.1(iii). This implies, as in the proof of Theorem 2.1(iii), that $\varphi_*(x) = 0$ for all $x \geq 0$ if $\mathbb{E}[Y_1] \leq 0$. □

4 Examples

Theorem 2.1 allows us to determine the optimal strategy and value function of the optimal underwriting problem for a wide class of examples. Regarding computing the objects appearing in Theorem 2.1, there are plenty of examples of spectrally negative Lévy processes where closed-form expressions exist for the scale function W_X, see e.g. Chapter 9 in [10] and the references therein. Otherwise, W_X and $\mathcal{A}_Y W_X$ can be computed by numerical Laplace inversion via (9) and (8) or by solving numerically renewal equations, recall (22) and note that W_X itself is the unique locally integrable solution to the renewal equation (12) with kernel $k(x) = \frac{\overline{\Pi}_X(x)}{c_X}$ and constant forcing function $f(x) = \frac{1}{c_X}$. In the rest of this section we will work out one example satisfying the conditions of the main theorem and cover an example that shows that the condition of the two hazard rates being ordered in Theorem 2.1(ii) is sharp. So an optimal strategy can consist of multiple switch levels and the earlier mentioned Example 1 in [3] suggests that there are cases where an optimal strategy must have infinitely many switch levels.

Example 4.1. We assume the tail Lévy measures of X and Y are respectively given by $\overline{\Pi}_X(x) = \lambda_X \mathrm{e}^{-r_X x}$ and $\overline{\Pi}_Y(x) = \lambda_Y \mathrm{e}^{-r_Y x}$ with $\lambda_X, \lambda_Y, r_X, r_Y > 0$. So the jump parts of X and Y are compound Poisson processes with exponentially distributed jumps. The hazard rates of Π_X and Π_Y are constants given by r_X and r_Y respectively and so they are decreasing and ordered. Hence, for any choice of the parameters $c_X, c_Y, \lambda_X, \lambda_Y, r_X, r_Y$, a 1-switch-level strategy (i.e. $Q^b(X,Y)$ or $Q^b(Y,X)$ for some $b \in [0,\infty]$) is optimal by Theorem 2.1. If $r_X = r_Y$, then by Proposition 2.2 we can conclude that if $\frac{\lambda_X}{c_X} \geq \frac{\lambda_Y}{c_Y}$ then an optimal strategy is to always be in mode Y whereas if $\frac{\lambda_X}{c_X} < \frac{\lambda_Y}{c_Y}$ then an optimal strategy is to always be in mode X. Since the case $r_X > r_Y$ can be dealt with by symmetry we assume without loss of generality $r_X < r_Y$ for the rest of the example. We further assume $\mathbb{E}[X_1] = c_X - \lambda_X/r_X > 0$ or $\mathbb{E}[Y_1] = c_Y - \lambda_Y/r_Y > 0$ so that ruin is not certain when the control is chosen optimally. For $\beta > \Phi_X(0) = \frac{\lambda_X}{c_X} - r_X \vee 0$,

$$\frac{1}{\beta\left(c_X - \frac{\lambda_X}{\beta + r_X}\right)} = \frac{\beta + r_X}{c_X\beta^2 + (c_X r_X - \lambda_X)\beta} = \begin{cases} \frac{1}{c_X\beta} + \frac{r_X}{c_X\beta^2} & \text{if } \mathbb{E}[X_1] = 0, \\ \frac{1}{c_X\mathbb{E}[X_1]}\left(\frac{c_X}{\beta} + \frac{\mathbb{E}[X_1] - c_X}{\beta + \frac{r_X}{c_X}\mathbb{E}[X_1]}\right) & \text{if } \mathbb{E}[X_1] \neq 0, \end{cases}$$

so by (9), for $x \geq 0$,

$$W_X(x) = \begin{cases} \frac{1}{c_X}\left(r_X x + 1\right) & \text{if } \mathbb{E}[X_1] = 0, \\ \frac{1}{c_X\mathbb{E}[X_1]}\left(c_X + (\mathbb{E}[X_1] - c_X)\mathrm{e}^{-\frac{r_X}{c_X}\mathbb{E}[X_1]x}\right) & \text{if } \mathbb{E}[X_1] \neq 0. \end{cases} \tag{24}$$

Of course $W_Y(x)$ can be expressed similarly. Hence, for $x > 0$, for both $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1] \neq 0$,

$$\mathcal{A}_Y W_X(x) = \frac{\lambda_X}{c_X^2}\left(c_Y - \frac{\lambda_Y}{r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1]}\right)\mathrm{e}^{-\frac{r_X}{c_X}\mathbb{E}[X_1]x} + \frac{\lambda_Y}{c_X}\left(\frac{\lambda_X}{c_X(r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1])} - 1\right)\mathrm{e}^{-r_Y x}.$$

Since $r_X < r_Y$ we know by Theorem 2.1(ii) that Q^{b^*} is optimal. Next we find explicit expressions for the optimal switching level b^*. Recall that we assume $\mathbb{E}[X_1] > 0$ or $\mathbb{E}[Y_1] > 0$.

- If $\frac{\lambda_X}{c_X} \geq \frac{\lambda_Y}{c_Y}$, then $\lim_{x\downarrow 0}\mathcal{A}_Y W_X(x) \geq 0$ and $\mathbb{E}[X_1] < \mathbb{E}[Y_1]$ so that $\mathbb{E}[Y_1] > 0$ and by Theorem 2.1(i), $b^* = 0$ and by Theorem 2.1(ii), $\varphi_*(x) = 1 - \frac{\lambda_Y}{c_Y r_Y}\mathrm{e}^{-\frac{r_Y}{c_Y}\mathbb{E}[Y_1]x}$, $x \geq 0$.
- If $\frac{\lambda_X}{c_X} < \frac{\lambda_Y}{c_Y}$ and $c_Y \leq \frac{\lambda_Y}{r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1]}$, then we must have $\mathbb{E}[X_1] > 0$ (as otherwise $\mathbb{E}[Y_1] \leq 0$ as well) and as $\frac{r_X}{c_X}\mathbb{E}[X_1] = r_X - \lambda_X/c_X < r_Y$, we have $\mathcal{A}_Y W_X(x) < 0$ for x sufficiently large, which implies by Theorem 2.1(i), $b^* = \infty$ and so $\varphi_*(x) = 1 - \frac{\lambda_X}{c_X r_X}\mathrm{e}^{-\frac{r_X}{c_X}\mathbb{E}[X_1]x}$, $x \geq 0$.
- If $\frac{\lambda_X}{c_X} < \frac{\lambda_Y}{c_Y}$ and $c_Y > \frac{\lambda_Y}{r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1]}$, then we must have $\mathbb{E}[Y_1] > 0$, $\lim_{x\downarrow 0}\mathcal{A}_Y W_X(x) < 0$ and $\mathcal{A}_Y W_X(x) > 0$ for x sufficiently large, which implies that b^* is the unique root of

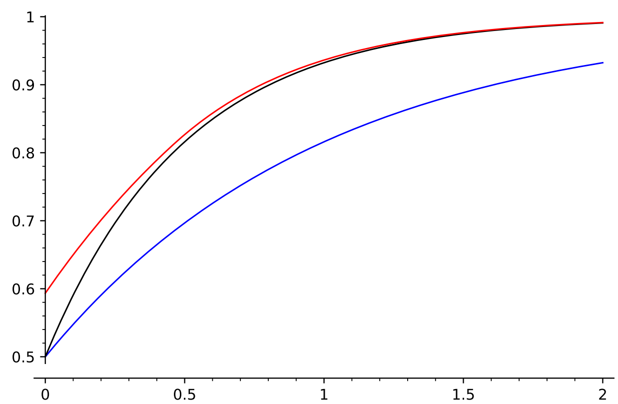

Figure 1: Displayed are three survival probabilities as a function of initial capital x associated with Example 4.1 for the parameter choices $c_X = c_Y = \lambda_X = 1$, $r_X = \lambda_Y = 2$ and $r_Y = 4$. Blue/bottom line: $\varphi_\infty(x)$ corresponding to always mode X; black/middle line: $\varphi_0(x)$ corresponding to always mode Y; red/top line: $\varphi_*(x) = \varphi_{b^*}(x)$ where $b^* = \frac{2}{3}\log(2)$ corresponding to optimal strategy.

$\mathcal{A}_Y W_X(x)$, $x \in (0,\infty)$, so

$$b^* = \frac{1}{\frac{r_X}{c_X}\mathbb{E}[X_1] - r_Y} \log\left(\frac{\frac{\lambda_X}{c_X}\left(c_Y - \frac{\lambda_Y}{r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1]}\right)}{\lambda_Y\left(1 - \frac{\lambda_X}{c_X(r_Y - \frac{r_X}{c_X}\mathbb{E}[X_1])}\right)} \right)$$

and an explicit expression for $\varphi_*(x) = \varphi_{b^*}(x)$ can be found via Theorem 2.1(ii) or (23). Figure 1 illustrates that there can be quite a significant advantage in having the ability to switch as opposed to always stick to mode X or always stick to mode Y, especially when initial capital is close to 0.

Example 4.2. Let X be as in Example 4.1 with $c_X = 1$, $\lambda_X = 29/10$ and $r_X = 52/10$. By Lemma 3.2 and (24) it readily follows that the survival probability corresponding to always choosing mode X is given by

$$\varphi_\infty(x) = \mathbb{E}[X_1]W_X(x) = 1 - \frac{29}{52}\mathrm{e}^{-23x/10} \quad \text{for } x \geq 0.$$

For the process Y, let $c_Y = 1$ and $\overline{\Pi}_Y(x) = \frac{1}{3}\mathrm{e}^{-3x} + \frac{8}{3}\mathrm{e}^{-6x}$, which implies $r_Y(x) = 3 + \frac{24}{\mathrm{e}^{3x}+8}$. It is easy to verify, via (9) and Lemma 3.2, that the survival probability corresponding to always choosing mode Y is given by

$$\varphi_0(x) = \mathbb{E}[Y_1]W_Y(x) = 1 - \frac{4}{9}\mathrm{e}^{-2x} - \frac{1}{9}\mathrm{e}^{-4x} \quad \text{for } x \geq 0.$$

Note that φ_∞ and φ_0 are not ordered in this case: we have that $\varphi_\infty(0) < \varphi_0(0)$ while for all x large enough $\varphi_\infty(x) > \varphi_0(x)$. Now, since $\mathbb{E}[X_1] > 0$, $\mathbb{E}[Y_1] > 0$ and both r_X and r_Y are decreasing, all assumptions of Theorem 2.1(ii) are satisfied with the single exception that r_X and r_Y are not ordered, as is easily verified. It turns out that the conclusion of Theorem 2.1(ii) does not hold in this example: none of the 1-switch-level strategies $Q^b(X,Y)$ and $Q^b(Y,X)$ for any $b \in [0,\infty]$ are optimal. To see this we eliminate them all. First, since φ_∞ and φ_0 are not ordered, it is immediately clear that the choices $b=0$ and $b=\infty$ cannot be optimal. Second, suppose that $Q^b(X,Y)$ were optimal for some $b \in (0,\infty)$ i.e. $\varphi_* = \varphi_b = \varphi_{Q^b(X,Y)}$. Denoting $\tau_b^-(Y) = \inf\{t>0 : Y_t < b\}$, we have for any $x > b$ that

$$\begin{aligned}
\varphi_*(x) &= \mathbb{E}_x\left[\mathbf{1}_{\{\tau_b^-(Y)=\infty\}} + \mathbf{1}_{\{\tau_b^-(Y)<\infty\}}\varphi_*\left(Y_{\tau_b^-(Y)}\right)\right] \\
&\leq \mathbb{E}_x\left[\mathbf{1}_{\{\tau_b^-(Y)=\infty\}} + \mathbf{1}_{\{\tau_b^-(Y)<\infty\}}\varphi_*(b)\right] \\
&= \varphi_0(x-b) + \varphi_*(b)\left(1-\varphi_0(x-b)\right) \\
&= 1 - \left(1-\varphi_*(b)\right)\left(1-\varphi_0(x-b)\right)
\end{aligned}$$

where the first equation uses the definition of $Q^b(X,Y)$ and the strong Markov property of the associated controlled process U^b and the inequality uses that φ_* is an increasing function due to optimality. It follows that, for $x > b$,

$$\frac{1-\varphi_*(x)}{1-\varphi_\infty(x)} \geq \left(1-\varphi_*(b)\right)\frac{1-\varphi_0(x-b)}{1-\varphi_\infty(x)}.$$

It is readily checked from the above expressions for φ_∞ and φ_0 that the right hand side of this inequality tends to ∞ as $x \to \infty$, and hence it follows that $\varphi_*(x) < \varphi_\infty(x)$ for all x large enough. However by optimality $\varphi_* \geq \varphi_\infty$ and we have arrived at a contradiction. Finally, suppose that $Q^b(Y,X)$ were optimal for some $b \in (0,\infty)$. Consider the strategy $\widehat{Q}^h$ informally described as picking mode X until the first time the controlled process goes strictly above h and afterwards follows the assumed optimal strategy $Q^b(Y,X)$, i.e. mode Y when the controlled process is below b and mode X when above b. We refrain from defining this strategy rigorously but this can be done in a similar way as for the strategy Q^b in Section 1. Recall the notation $\tau_h^+(Z) = \inf\{t>0 : Z_t > h\}$ and $\tau_0^-(Z) = \inf\{t>0 : Z_t < 0\}$ for a process Z. By the strong Markov property of the controlled processes associated with $Q^b(Y,X)$ and $\widehat{Q}^h$ and since a non-trivial Lévy process does not stay in any finite interval forever, we have for any $0 < h < b$,

$$\varphi_*(0) = \varphi_*(h)\mathbb{P}_0(\tau_h^+(Y) < \tau_0^-(Y)) \quad \text{and} \quad \varphi_{\widehat{Q}^h}(0) = \varphi_*(h)\mathbb{P}_0(\tau_h^+(X) < \tau_0^-(X)).$$

For a spectrally negative Lévy process Z with $\overline{\Pi}_Z$ continuous we have by (16) and a Taylor approximation,

$$\mathbb{P}_0(\tau_h^+(Z) < \tau_0^-(Z)) = \frac{W_Z(0)}{W_Z(h)} = \frac{1}{1 + \frac{\overline{\Pi}_Z(0)}{c_Z}h + o(h)},$$

where $o(h)$ is a function such that $\frac{o(h)}{h} \to 0$ as $h \downarrow 0$ and where we used that $W_Z(0) = 1/c_Z$ and $\lim_{x\downarrow 0} W_Z'(x) = \overline{\Pi}_Z(0)/c_Z^2$ where the latter follows from the former, (10) and (7). Since $\frac{\overline{\Pi}_X(0)}{c_X} < \frac{\overline{\Pi}_Y(0)}{c_Y}$, we can conclude that for all $h > 0$ small enough $\mathbb{P}_0(\tau_h^+(X) < \tau_0^-(X)) > \mathbb{P}_0(\tau_h^+(Y) < \tau_0^-(Y))$ and hence $\varphi_{\widehat{Q}^h}(0) > \varphi_*(0)$, which is again a contradiction.

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