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TESI DI LAUREA

Coomologia orbifold dello spazio dei moduli delle curve ellittiche con punti marcati

Relatore: Ch.ma Prof. Barbara Fantechi

Laureando: Nicola Pagani

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Introduzione

My task which I am trying to achieve is, by the power of the written word, to make you hear, to make you feel – it is, before all, to make you see*. That – and no more, and it is everything. If I succeed, you shall find there according to your deserts: encouragement, consolation, fear, charm – all you demand; and, perhaps, also that glimpse of truth for which you have forgotten to ask.*
Joseph Conrad, "The Nigger of the Narcissus", preface.

1 Spazi di moduli

In questo lavoro studiamo spazi i cui punti rappresentano, o parametrizzano, classi di isomorfismo di oggetti geometrici: questi spazi sono chiamati spazi di moduli. Esempi classici e molto studiati in geometria algebrica classica di spazi di moduli sono gli spazi proiettivi, più in generale le Grassmaniane e le varietà di bandiera. L'idea di costruire spazi i cui punti parametrizzano oggetti geometrici modulo isomorfismo può essere naturalmente estesa a oggetti geometrici in un'accezione più generale (oggetti che non siano necessariamente varietà algebriche) e il loro studio può contribuire a fornire sugli enti geometrici di partenza informazioni che non sono direttamente osservabili sulla base di un loro studio caso per caso. Proprietà come la connessione, la irriducibilità, la singolarità si traducono per gli oggetti geometrici studiati nel fatto che, ad esempio, non si possa passare mediante deformazioni dall'uno all'altro oppure nel fatto che per deformare un certo spazio in un altro, si deve passare inevitabilmente per un terzo.

Il primo passo da fare per uno studio di tali spazi di moduli in geometria algebrica è naturalmente dotare gli stessi di una struttura algebro-geometrica. Il nome "spazio di moduli" è dovuto a Riemann, che con il termine moduli indicava genericamente i parametri continui da cui dipende una curva. Nel 1857 egli calcolò, in un modo che oggi definiremmo intuitivo, la dimensione di questo spazio, che per le curve di genere g è $3g-3$ se $g \geq 2$, uno se $g=1$ e zero se $g=0$, dando per scontata la struttura algebro-geometrica dello spazio dei moduli, ritenuta in qualche modo ovvia. Con un approccio analitico al problema, Klein dimostrò che lo spazio delle curve algebriche di genere g è irriducibile, utilizzando mappe su rivestimenti n-upli di $\mathbb{P}^1$. Anche Hurwitz seguì un metodo analogo, studiando le varietà dei moduli di tali rivestimenti di genere g e mappe da essi nello spazio dei moduli delle curve di genere g. Nei primi anni del XX secolo Enriques e Severi furono interessati, marginalmente il primo, più sistematicamente il secondo, al problema di produrre una dimostrazione puramente algebrico-geometrica dell'irriducibilità dello spazio dei moduli delle curve di genere fissato. Entrambi ritenevano di potersi restringere allo studio delle classi d'isomorfismo delle curve piane con singolarità ordinarie, di genere fissato. La loro dimostrazione risultò incompleta, e solo nel 1969 il celebre lavoro di Deligne e Mumford riuscì a produrre una dimostrazione della irriducibilità di questo spazio di moduli, basandosi tuttavia su risultati classici topologici. In questo lavoro, Deligne e Mumford affrontavano il problema dotando in due modi differenti questo spazio di moduli di struttura algebrica: dandogli la struttura di varietà algebrica mediante tecniche di Teoria Geometrica degli Invarianti, e dotandolo della struttura di stack algebrico, proponendo in quello stesso lavoro la parola stack come traduzione dal francese *champ*. La parola *champ* era stata usata per la prima volta da Giraud nel suo libro *Cohomologie non abelienne*, nel quale l'autore sviluppava tra le altre la nozione di stack, la cui invenzione possiamo in ultima analisi ricondurre a Grothendieck. La nozione di stack algebrico, oggi usato come strumento per dotare di struttura algebrica spazi di moduli di molteplici oggetti geometrici, è per nascita pertanto intimamente legata agli spazi di moduli di curve algebriche. La prima delle due costruzioni di Deligne e Mumford produce una varietà quasiproiettiva singolare, che tuttavia vedremo non essere una famiglia universale. In altre parole il funtore di moduli (che verrà definito in seguito) non risulta rappresentabile da questa varietà quasiproiettiva ma è solamente corappresentabile. Si parla in questo caso di spazio dei moduli grossolano. La seconda costruzione produce invece uno stack algebrico liscio (detto anche orbifold), che invece riesce a rappresentare il funtore di moduli, e per questo viene detto spazio dei moduli fine per il problema di moduli. Come vedremo meglio in seguito, responsabile per la non esistenza di una varietà algebrica che sia uno spazio dei moduli fine è la presenza di un gruppo di automorfismi non banale per alcune curve proiettive. Nel corso degli anni sono state proposte diverse soluzioni per ovviare a questo inconveniente. Innanzitutto si è pensato di escludere le curve con gruppo degli automorfismi non banale e di studiare l'insieme che parametrizza queste ultime. Un altro approccio ha invece portato ad aggiungere struttura alle curve, imponendo che abbiano dei punti fissati. Un passo ulteriore nello studio di questi spazi è una loro compattificazione: si tratta di aggiungere punti allo spazio dei moduli delle curve di genere fissato che possano interpretarsi come punti limite. La scelta giusta risulta nelle classi di isomorfismo di famiglie di curve nodali con determinate proprietà di intersezione.

L'idea di base degli stack è quella di sostituire l'insieme delle classi d'isomorfismo di famiglie di curve con il gruppoide che ha per oggetti le famiglie e per morfismi gli isomorfismi di famiglie. Le condizioni a cui il gruppoide deve soddisfare per essere uno stack sono analoghe alle condizioni di fascio per un funtore. Sempre nel lavoro di Deligne e Mumford si dimostra che gli stack dei moduli delle curve di genere fissato con n punti fissati sono algebrici, i.e. localmente isomorfi a schemi (in una topologia, quella étale, che verrà introdotta più avanti, e che è più fine di quella di Zariski).

2 La coomologia orbifold

Negli ultimi 20 anni gli stack algebrici sono stati molto studiati, anche in ambito fisico. La motivazione per la geometria e topologia degli orbifold in ambito fisico viene dalla teoria delle stringhe sugli orbifold, annunciata dai fisici Dixon, Harvey, Vafa, Witten nel 1985 [DHVW]. Gli orbifold sono stati studiati in matematica dagli anni cinquanta, come estensione della teoria delle varietà lisce. La geometria e topologia "stringy" degli orbifold ha una natura diversa: il suo scopo è studiare proprietà degli orbifold legate alle stringhe. Negli ultimi anni la teoria delle stringhe su orbifold è diventata molto popolare in ambito fisico, come dimostra anche il numero di lavori che sono stati pubblicati su questo argomento. Contemporaneamente la sua controparte matematica ha avuto numerosi sviluppi con l'inizio del nuovo millennio.

Una nuova descrizione degli orbifold nasce nella geometria algebrica a partire dai problemi di moduli di curve con punti marcati: essi vengono visti in questo caso come stack algebrici lisci di Deligne-Mumford. In questa nuova descrizione la struttura di 2-categoria risulta naturalmente dalla costruzione, laddove la nozione di morfismo fra orbifold nella descrizione precedente aveva richiesto un lungo travaglio nella comunità scientifica, come dimostra il fatto che il primo lavoro pubblicato di

Satake sui V-manifold [Sa] propone una definizione di morfismo di orbifold non più utilizzata.

Nel lavoro [DHVW] si studia un orbifold quoziente globale di dimensione reale 6 compatta. Gli autori osservano inizialmente che un teorema dell'indice (una versione orbifold della formula della traccia di Lefschetz ([B], [GH] 3.4)) porta ad un legame fra la caratteristica di Eulero topologica della varietà e la traccia di un certo operatore di interesse fisico che rispetta certe condizioni al contorno di periodicità. Per studiare il caso dell'orbifold, si introducono condizioni al contorno "twistate", che tengano cioè conto delle identificazioni date dall'azione del gruppo. Portando avanti i conti per la traccia dell'operatore con le nuove condizioni adattate al caso dell'orbifold, al fine di preservare la precedente uguaglianza anche sull'orbifold, gli autori suggeriscono una nuova definizione per la caratteristica di Eulero dell'orbifold, che tenga conto dell'azione del gruppo e non solo della sua struttura topologica. Questa caratteristica di Eulero per orbifold soddisfa la proprietà di essere uguale alla caratteristica di Eulero usuale di una sua "buona" risoluzione di singolarità (risoluzione crepante). Successivamente sono stati studiati altri invarianti coomologici come i numeri di Hodge per orbifold, sempre con l'idea che questi dovessero coincidere con una risoluzione crepante delle singolarità dell'orbifold (questo fatto è stato dimostrato nel 1996 in un lavoro di Batyrev-Borisov [BB]).

Seguendo il fondamentale impulso dato da questo lavoro fisico, Chen e Ruan danno una definizione in [CR1], [CR2] e [CR3] di tutta la coomologia per il caso degli orbifold, con l'idea che questo possa introdurre una nuova geometria e una nuova topologia per gli orbifold, il cui centro risieda nel concetto di settore twistato. In questa definizione la caratteristica di Eulero e i numeri di Hodge coincidono con le quantità predette sulla base di considerazioni di carattere fisico. Questa definizione nasce pertanto insieme ad una "congettura crepante", secondo la quale la (struttura additiva della) coomologia orbifold è isomorfa alla coomologia standard di una sua risoluzione "minimale" (crepante). Ancora oggi per riferirsi a questo tipo di coomologia, la letteratura utilizza il termine "stringy cohomology". Questa congettura è stata verificata nel nuovo contesto della definizione di Chen e Ruan, e nel caso dei numeri di Hodge contemporaneamente nei lavori di Lupercio-Poddar [LP] e Yasuda [Y] del 2003.

Si osserva poi che, come nel caso delle manifold, la coomologia è naturalmente dotata del prodotto cup, alla struttura additiva della coomologia orbifold si può associare in modo naturale un prodotto, questi risultano essere la parte di grado 0 della small quantum cohomology.

Successivamente nel lavoro [AGV] gli autori propongono una nuova definizione nel contesto della definizione di orbifold come stack algebrico liscio di Deligne e Mumford.

La definizione è stata poi estesa nel 2001 contemporaneamente in un lavoro di Fantechi-Göttsche [FG] e, (nel caso degli stacks che sono prodotti simmetrici) in un lavoro di Uribe, al caso non commutativo, per le orbifold che sono quozienti globali.

Cito ora di seguito i lavori apparsi a seguito della definizione di Chen e Ruan in cui viene calcolata la orbifold (stringy) cohomology. Per gli orbifold abeliani, e più in generale per i quozienti arbitrari di un toro, i lavori di [CH] [GHK]. Per gli stack torici, il lavoro di Yunfeng Jiang [J]. Nel lavoro di Bryan-Graber-Pandharipande [BGP] si calcola l'intera quantum cohomology per il caso $\mathbb{C}^2/Z_3$. Perroni [Pe] calcola per intero la coomologia orbifold per le orbifold con singolarità di tipo A_n, i settori twistati e l'età per le singolarità di tipo D ed E. Infine per gli spazi proiettivi pesati i lavori di Mann [M] e quello molto recente di Coates-Corti-Lee-Tseng [CCLT]. Questi ultimi ne hanno calcolato l'intera quantum cohomology. Infine nel lavoro [GH] si calcola la coomologia orbifold per le varietà ipertoriche. I lavori [BCS] e [JT] calcolano l'anello di Chow (classi di coomologia algebriche) per stacks torici e ipertorici.

3 Contenuto della tesi

La coomologia orbifold è definita dal 2000 ma fino ad ora è stata calcolata su relativamente pochi esempi. L'idea da cui prende le mosse questa tesi è appunto calcolarla sui primi stack algebrici: gli spazi di moduli di curve di genere fissato con punti marcati. Il caso di genere 0, non presenta problematiche legate ad automorfismi non banali, come vedremo meglio in seguito. Lo spazio dei moduli delle curve lisce di genere 0 con n punti marcati risulta essere una varietà quasiproiettiva, la sua compattificazione (le curve di genere 0 nodali) è una varietà proiettiva, pertanto la coomologia orbifold coincide con la usuale coomologia singolare, che a sua volta coincide con l'anello di Chow, calcolato in [K] da Sean Keel nel 1992. Il primo caso non banale risulta pertanto quello delle curve di genere 1. La coomologia usuale dello spazio dei moduli delle curve ellittiche lisce e di quelle nodali stabili è stata molto studiata già dai lavori di Harer del 1982. Negli anni '90 il problema è stato affrontato seguendo tecniche molto differenti, ma senza riuscire a produrre un risultato definitivo.

In questo lavoro, dopo aver osservato che la coomologia orbifold dello spazio dei moduli delle curve lisce e nodali ellittiche (stabili) dipende dalla sua coomologia usuale, daremo una descrizione completa dei settori twistati dello stack d'inerzia di questi due spazi dei moduli, e ridurremo lo studio della coomologia orbifold allo studio della coomologia usuale. Della coomologia usuale è completamente nota la struttura di spazio vettoriale, mentre il prodotto è stato molto studiato negli ultimi quindici anni, con risultati definitivi soltanto per i casi fino a 4 punti marcati.

Il risultato principale di questo lavoro pertanto è la descrizione dei settori twistati dello stack d'inerzia dello spazio dei moduli delle curve ellittiche lisce e nodali al variare dei punti marcati. Questo, grazie ai risultati di Keel e alla formula di Kunneth fornisce la dimensione della coomologia dei settori twistati.

Rimangono ancora da studiare la graduazione della coomologia, che nella notazione introdotta da Miles Reid, prende il nome di età, e che viene anche detta shift fermionico, e infine il prodotto. Per l'età forniamo risultati parziali, che comprendono la descrizione completa del risultato liscio e la descrizione dell'età per il settore twistato nel caso stabile relativo all'automorfismo (-1).

4 Descrizione delle singole sezioni

Nell'appendice A si introducono gli stacks e gli stacks algebrici, attraverso il linguaggio degli pseudofuntori. Alla fine del primo capitolo viene svolto nel dettaglio l'esempio degli stack algebrici che sono quozienti globali.

Nell'appendice B restringiamo la generalità del discorso agli stack algebrici lisci, che sono una nozione equivalente ad una già diffusa in geometria differenziale negli anni '50: quella di orbifold (anche detti V-manifold), nel nostro caso gli orbifolds saranno non necessariamente ridotti. Si mostrerà un cenno della dimostrazione dell'equivalenza delle due nozioni. Nello stesso capitolo introduciamo la coomologia e la coomologia orbifold di uno stack nella definizione data da Chen e Ruan, con qualche semplificazione legata al fatto che tutti i gruppi di automorfismi che compaiono in questo lavoro sono gruppi commutativi.

Nell'appendice C si presenta una review di curve, si focalizza l'attenzione sulle curve di genere 1 e si introducono le nozioni fondamentali di famiglia di curve e famiglia universale, per arrivare fino a dotare gli spazi di moduli di curve di genere fissato e con punti marcati della struttura di stack algebrico. L'ultima sezione presenta una descrizione informale del teorema di Knudsen sulla curva universale.

L'ultima appendice presenta i risultati originali di questa tesi: la descrizione esplicita dei settori twistati nel caso non compatto (più semplice) e nel caso compatto. Infine si danno risultati parziali sul calcolo dello shift fermionico (età).

5 Notazioni

Gli schemi di cui si parla in questo lavoro sono tutti di tipo finito sopra un campo $\mathbb{K}$ algebricamente chiuso e di caratteristica 0. I risultati ottenuti sono tutti sul campo dei numeri complessi, tuttavia in alcune sezioni precedenti ho volutamente omesso la sostituzione $\mathbb{K} := \mathbb{C}$ per mantenere la stessa notazione dei testi cui mi riferivo. I punti s di uno schema S sono identificati con i morfismi i :Spec$(K(s)) \longrightarrow S$ dove $K(s)$ è il campo residuo dell'anello locale in s di S.

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La coomologia orbifold di $\mathcal{M}_{1,n}$ e $\overline{\mathcal{M}}_{1,n}$

Rimandiamo all'appendice per tutti i dettagli, e in questo paragrafo forniamo una concisa presentazione del risultato ottenuto.

La nozione di stack algebrico è presentata nell'appendice A, nell'appendice B si introduce la nozione di coomologia orbifold per uno stack algebrico liscio, come coomologia singolare del suo stack d'inerzia. Nell'appendice C si mostra come $\mathcal{M}_{1,n}$ e $\overline{\mathcal{M}}_{1,n}$ siano in modo naturale stack algebrici. Per le notazioni sugli spazi rimandiamo all'appendice D. I risultati principali ottenuti in questo lavoro, sono i seguenti:

Teorema 1. *(Appendix D; 1.19, 1.20, 1.21, 1.22, [Co3]) La struttura additiva della coomologia orbifold a coefficienti razionali di $\mathcal{M}_{1,n}$ e di $\overline{\mathcal{M}}_{1,n}$ è nota.*

Teorema 2. *I settori twistati dello stack d'inerzia di $\mathcal{M}_{1,n}$ sono unione disgiunta di stacks i cui spazi coarse soggiacenti sono punti, la retta proiettiva meno un numero finito di punti, e rivestimenti ramificati su quest'ultima (i.e. ancora la retta proiettiva senza un numero finito di punti). Più precisamente, vale la seguente descrizione caso per caso:*

- *(Lo stack d'inerzia di $\mathcal{M}_{1,2}$)*

$$I(\mathcal{M}_{1,2}) = (\mathcal{M}_{1,2}, 1) \coprod (A_1, -1) \coprod (C_4, i/-i) \coprod (C_6, \epsilon^2/\epsilon^4)$$

- *(Lo stack d'inerzia di $\mathcal{M}_{1,3}$)*

$$I(\mathcal{M}_{1,3}) = (\mathcal{M}_{1,3}, 1) \coprod (A_2, -1) \coprod (C_6, \epsilon^2/\epsilon^4)$$

- *(Lo stack d'inerzia di $\mathcal{M}_{1,4}$)*

$$I(\mathcal{M}_{1,4}) = (\mathcal{M}_{1,4}, 1) \coprod (A_3, -1)$$

- *Lo stack d'inerzia di $\mathcal{M}_{1,n}$, quando $n > 4$, è isomorfo a $\mathcal{M}_{1,n}$.*

Nel seguito, per semplicità di notazione, assumiamo che $\overline{\mathcal{M}}_{0,1}$ e $\overline{\mathcal{M}}_{0,2}$ siano un punto (mentre secondo la definizione standard dovrebbero essere l'insieme vuoto) e che $(0,0)!$ sia 0 invece che 1.

Teorema 3. *Gli spazi coarse dei settori twistati dello stack d'inerzia di $\overline{\mathcal{M}}_{1,n}$ sono, a meno di isomorfismo, unione disgiunta di prodotti di $\overline{\mathcal{M}}_{0,k}$. Più precisamente vale la seguente descrizione dei settori twistati dello stack d'inerzia:*

1.

$$I(\overline{\mathcal{M}}_{1,2}) = (\overline{\mathcal{M}}_{1,2}, 1) \coprod (\overline{\mathcal{M}}_{1,1}, -1) \coprod (\overline{A_1}, -1) \coprod 2\ (C_4, i/-i) \coprod (C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5) \coprod$$
$$\coprod (C_6, \epsilon^2/\epsilon^4)$$

2.

$$I(\overline{\mathcal{M}}_{1,3}) = (\overline{\mathcal{M}}_{1,3}, 1) \coprod (\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4}, -1) \coprod 3\ (\overline{A_1}, -1) \coprod (\overline{A_2}, -1) \coprod$$
$$\coprod 2(C_4, i/-i) \coprod (C_4 \times \overline{\mathcal{M}}_{0,4}, i/-i) \coprod (C_6 \times \overline{\mathcal{M}}_{0,4}, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5) \coprod 4(C_6, \epsilon^2/\epsilon^4)$$

3.

$$I(\overline{\mathcal{M}}_{1,n}) = \overline{\mathcal{M}}_{1,n} \coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1, \alpha_2)!\ \overline{A_1} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$
$$\coprod_{\alpha_1+\alpha_2=n-3,\ \alpha_i\geq 0} ((\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2))!\ \overline{A_2} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$
$$\coprod_{\alpha_1+\alpha_2=n-4,\ \alpha_i\geq 0} (2(\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2)!)\ \overline{A_3} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2} \coprod \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1} \coprod$$
$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1, \alpha_2)!\ \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2} \coprod \overline{\mathcal{M}}_{0,n+1} \coprod$$

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1,\alpha_2)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\times\overline{\mathcal{M}}_{0,\alpha_2+2}\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\coprod_{\alpha_1=0}^{n-2}(n-1-\alpha_1,\alpha_1)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\ \times$$

$$\times\left(\coprod_{\beta_1+\beta_2=n-1-\alpha_1,\ \beta_i\geq 0,\ \beta_1>\beta_2}(n-1-\alpha_1-\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_2+1}\coprod\right.$$

$$\left.\coprod_{2\beta_1=n-1-\alpha_1}\frac{1}{2}(\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_1+1}\right)\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\coprod_{\alpha_1=0}^{n-2}(n-1-\alpha_1,\alpha_1)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\ \times$$

$$\times\left(\coprod_{\beta_1+\beta_2=n-1-\alpha_1,\ \beta_i\geq 0,\ \beta_1>\beta_2}(n-1-\alpha_1-\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_2+1}\coprod\right.$$

$$\left.\coprod_{2\beta_1=n-1-\alpha_1}\frac{1}{2}(\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_1+1}\right)\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\overline{\mathcal{M}}_{0,n+1}\coprod\overline{\mathcal{M}}_{0,n+1}$$

Teorema 4. *(Appendix D; Section 2) La struttura di spazio vettoriale graduato sui razionali della coomologia orbifold razionale di $\mathcal{M}_{1,n}$ è nota.*

Infine forniamo risultati parziali per la graduazione della coomologia orbifold razionale di $\overline{\mathcal{M}}_{1,n}$.

Appendix A

Stacks and Algebraic Stacks

References for this section are [G], [Vi1], [Vi2]. There are at least three ways to define stacks: as the datum of a 2-functor, of a pseudofunctor in groupoids, or of a fibered category in groupoids. In each cases you need a notion of topology for a category, namely a Grothendieck topology. The third construction is equivalent to the first two once you assumes a form of the axiom of choice, which allows you to have a well-defined, unique pull-back. The first two definitions are equivalent once you write down all the axioms for a general 2-category and for a general 2-functor and finally observing that a 1-category is in a canonical way a 2-category (with only identities as 2-morphisms). This is observed with much more detail for example in [G]. The definition of stack as a 2-functor needs some acquaintance with 2-categories, while the definition which uses pseudofunctors in groupoids requires only to understand the structure of the 2-category of groupoids, and so makes possible to write a definition *ad hoc* if one wants to develop the theory just to give a definition of stack.

1 Pseudofunctors

In this section $\mathcal{C}$ is a fixed category, while $\mathcal{GPD}$ is the category of groupoids, whose objects are groupoids and whose morphisms are homomorphism between groupoids.

Definition 1.1. A *groupoid* is a small category whose morphisms are all isomorphisms.

Hence the category of all groupoids has an additional structure. In fact the set $Hom(A, B)$ where A, B are two groupoids, is actually a category, whose morphisms are natural equivalence between functors from A to B. This gives the category of groupoids a structure of 2-category. Anyway, we won't enter the details of 2-categories. For a reference, see [G] [Bo]. We stress that our definition is equivalent to the algebraic one:

Remark 1.2. A *groupoid* with base B is a set G with mappings α and β from G onto B and a partially defined binary operation $(g,h) \longrightarrow gh$, satisfying the following four conditions:

- gh is defined whenever $\beta(g) = \alpha(h)$ and in this case $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$.
- if either of $(gh)k$ and $g(hk)$ are defined so is the other and they are equal.
- for each $g \in G$ there are two identity elements λ_g and ρ_g satisfying $\lambda_g g = g = g\rho_g$.
- for each $g \in G$ there is an inverse element g^{-1} satisfying $gg^{-1} = \lambda_g$ and $g^{-1}g = \rho_g$.

If G is a groupoid from this last viewpoint, it is a category whose objects are the elements of B, and whose morphisms are all the products in the groupoid of the form m_g, where $g \in G$. If m_g is a morphism: $g' \longrightarrow g''$, since G is a groupoid, $m_{g^{-1}} : g'' \longrightarrow g'$ is its inverse. Following this viewpoint, an homomorphism between two groupoids G and G' is exactly a functor between the data of the two groupoids seen as categories.

Since all morphisms are isomorphisms, all natural transformations between functors (i.e. groupoids' homomorphisms) give a natural isomorphism between the two functors.

Definition 1.3 (cf. [Vi2] pag.50)**.** A contravariant *pseudofunctor* F in groupoids:

$$F : \mathcal{C} \longrightarrow \mathcal{GPD}$$

consists of the following data:

- For each C object of $\mathcal{C}$, a groupoid $F(C)$;
- For each arrow $\phi : C \longrightarrow D$, a morphism of groupoids $F(\phi) : F(D) \longrightarrow F(C)$;

- For each object C of $\mathcal{C}$, an isomorphism $\epsilon_C : F(Id_C) \longrightarrow Id_{F(C)}$;
- For each $S \xrightarrow{\phi} T \xrightarrow{\psi} U$ an isomorphism between functors $\alpha_{\phi,\psi} : F(\phi) \circ F(\psi) \Rightarrow F(\psi \circ \phi)$

satisfying the following conditions:

1. If $\phi : C \longrightarrow D$ is an arrow in $\mathcal{C}$, these two equalities hold:

$$\alpha_{Id_C,\phi} = \epsilon_C \circ \eta_{F(\phi)}$$

$$\alpha_{\phi,Id_V} = \eta_{F(\phi)} \circ \epsilon_D$$

   where $\eta_F(\phi)$ is the natural transformation associated to the funtor $F(\phi)$.

2. Whenever there are arrows $S \xrightarrow{\phi} T \xrightarrow{\psi} U \xrightarrow{\lambda} V$ in $\mathcal{C}$, the following diagram of functors and natural transformations commutes:

$$\begin{tikzcd}
F(\phi) \circ F(\psi) \circ F(\lambda) \arrow[r, Rightarrow, "\alpha_{\phi,\psi} \times Id"] \arrow[d, Rightarrow, "Id \times \alpha_{\psi,\lambda}"'] & F(\psi \circ \phi) \circ F(\lambda) \arrow[d, Rightarrow, "\alpha_{\psi\circ\phi,\lambda}"] \\
F(\phi) \circ F(\lambda \circ \psi) \arrow[r, Rightarrow, "\alpha_{\phi,\lambda\circ\psi}"] & F(\lambda \circ \psi \circ \phi)
\end{tikzcd}$$

Remark 1.4. In the article [Vi2] pag.49, the author points out that there is no real lack of information asking, instead of the third condition, directly that $F(Id_C) = Id_{F(C)}$. In [G] (pag.29-30), the given definition of pseudofunctor takes in account this kind of simplification.

Example 1.5. First of all, observe that there is an obvious fully faithful functor $O : \mathcal{SET} \longrightarrow \mathcal{GPD}$ (remember the latter category is the category of all groupoids). If S is a set, $O(S)$ is the groupoid whose elements are the elements of the set, and whose morphisms are just the identities. Let C be an object in $\mathcal{C}$. The Yoneda functor:

$$h_C : \mathcal{C}^{op} \longrightarrow \mathcal{SET}$$

defined as $h_C(X) = Hom(X, C)$, $h_C(f) = Hom(f, C)$ is a functor from $\mathcal{C}$ to $\mathcal{SET}$. So, $O \circ h_X$ gives a (contravariant) functor from $\mathcal{C}$ to groupoids. A functor in groupoids is a very special case of a pseudofunctor in groupoids, where all the natural transformations are just the identities. This construction is very useful because it gives an embedding of our category in the larger category $Hom(\mathcal{C}^{op}, \mathcal{SET})$, where objects are functors and morphisms are natural trasformation of functors. Composing with the obvious O functor, it embeds the category C in the category of pseudofunctors in groupoids (in fact, in the 2-category of pseudofunctors). Yoneda's lemma tells exactly that the functor h is fully faithful.

Lemma 1.6 (Yoneda)**.** *In the previous notations, if $F : \mathcal{C} \longrightarrow \mathcal{SET}$ is a functor, there is a canonical bijection:*

$$Hom(h_X, F) \cong F(X)$$

In the particular case where $F = h_Y$, you get exactly that the functor h is fully faithful.

Proof. Given a natural transformation $\tau : h_X \longrightarrow F$, one gets an element $\xi \in F(X)$, defined as the image of the identity map $Id_X \in h_X(X)$ via the function $\tau_X : h_X(X) \longrightarrow F(X)$. Conversely, given an element $\xi \in F(X)$, one can define a morphism $\tau : h_X \longrightarrow F$ as follows. Given an object U of $\mathcal{C}$, an element of $h_X(U)$ is an arrow $f : U \longrightarrow X$. This arrow induces a function $F(f) : F(X) \longrightarrow F(U)$. Then one defines a function $\tau_U : h_X(U) \longrightarrow F(U)$ by sending $F \in h_X(U)$ to $F(f(\xi)) \in F(U)$. It is straightforward to check that the τ defined is in fact a morphism. The two maps $\tau \longrightarrow \xi$ and $\xi \longrightarrow \tau$ are clearly inverse each to the other. □

Definition 1.7. A *representable functor* on the category $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \longrightarrow \mathcal{SET}$$

which is isomorphic to a functor of the form h_X for some object X of $\mathcal{C}$. We will say that X *represents* F.

The Yoneda lemma guarantees that two objects representing the same functor are canonically isomorphic. The Yoneda embedding allows us to denote by X the functor h_X. Since no possible confusion arises, this substitution will be sometimes done.

Definition 1.8. A *universal object* for F is a pair (X, ξ) consisting of an object X of $\mathcal{C}$, and an element $\xi \in F(X)$ s.t. for each object U in $\mathcal{C}$ and each $\sigma \in F(U)$, there is an unique arrow $f : U \longrightarrow X$ such that $F(f(\xi)) = \sigma \in F(U)$.

The pair (X, ξ) is a universal object if the morphism $h_X \longrightarrow F$ defined by ξ is an isomorphism. Since every natural transformation $h_X \longrightarrow F$ is defined by some object $\xi \in F(X)$ we get that F is representable if and only if it has a universal object.

Definition 1.9. A *morphism of pseudofunctors in groupoids* consists of the following data:

- For every object $A \in \mathcal{C}$, a groupoid homomorphism $P_A : F(A) \longrightarrow G(A)$.
- For every morphism $f : B \longrightarrow A$, a morphism of functors between groupoids $\phi_f : P_A \circ F(f) \longrightarrow G(f) \circ P_B$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "P_A"'] & F(B) \arrow[d, "P_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

Such that $\phi_f \circ \phi_g = \phi_{g \circ f}$ for every pair of composable morphisms f, g in $\mathcal{C}$. An *isomorphism of pseudofunctors in groupoids* is a morphism $P : F \longrightarrow G$ such that there exists one other morphism $Q : G \longrightarrow F$ such that $PQ \cong \mathrm{Id}_G$ and $QP \cong \mathrm{Id}_F$.

Equivalently, a morphism is an isomorphism if for every object A of $\mathcal{C}$, P_A is an equivalence of groupoids (i.e. a fully faithful functor which is essentially surjective).

Remark 1.10. The standard literature defines a stack as a fibered category in groupoids. In such a language, the definition of morphism is much simpler [[Vi2] Def.3.6]. For a proof of the equivalence between the two approaches, once one assume the axiom of choice, see [[Vi2] 3.1.2,3.1.3].

Lemma 1.11 (Weak 2-Yoneda lemma, [Vi2] 3.6.1)**.** *The function that sends each arrow $f : X \longrightarrow Y$ to the corresponding morphism of pseudofunctors $h_f : h_X \longrightarrow h_Y$ is a natural isomorphism of groupoids.*

Definition 1.12. A pseudofunctor is said *representable* if it is isomorphic to a pseudofunctor of the form h_X for some $X \in \mathcal{C}$.

Remark 1.13. As in the case of functors, there is a stronger version of the 2-categorical Yoneda lemma. There is an isomorphism of groupoids:

$$\mathrm{Hom}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$$

which is defined sending a morphism of pseudofunctors $\alpha : h_X \longrightarrow F$ to $\alpha(Id_X)$, where $Id_X \in h_X(X)$.

Till now, we have embedded our category in the category of pseudofunctors (which can be seen also as presheaves in groupoids, since a presheaf of sets is nothing but a contravariant functor in $\mathcal{SET}$). Stacks are more than just presheaves, actually they are sheaves.

2 Grothendieck topology

Recall, that if X is a topological space, a sheaf of sets on it is a contravariant functor on $\mathcal{TOP}_X$, the category whose objects are the open subsets of X and whose morphisms are just inclusions (presheaf), satisfying the exactness of the following diagram:

$$F(U) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

for every U object in $\mathcal{TOP}_X$ and for every covering $(U_\alpha \longrightarrow U)$ of U. In this context the fiber product is just a fancy way to write down intersection. In a Grothendieck topology the open sets of a space are maps into this space; instead of intersection we have to look at fibered products.

Definition 2.1 ([Vi2], pg.28)**.** Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ of a collection of sets of arrows $\{U_i \longrightarrow U\}$ called *coverings* of U, so that the following conditions are satisfied:

1. If $V \longrightarrow U$ is an isomorphism, then the set $\{V \longrightarrow U\}$ is a covering.

2. If $\{U_i \longrightarrow U\}$ is a covering and $V \longrightarrow U$ is an arrow, then the fibered products $\{U_i \times_U V\}$ exist, and the collection of projections $\{U_i \times_U V \longrightarrow V\}$ is a covering.

3. If $\{U_i \longrightarrow U\}$ is a covering, and for each index i $\{V_{ij} \longrightarrow U\}$ is a covering (here j varies in a set indexed on i), the collection of composites $\{V_{ij} \longrightarrow U_i \longrightarrow U\}$ is a covering of U.

Remark 2.2. Notice that if $\{U_i \longrightarrow U\}$ and $\{V_j \longrightarrow U\}$ are two coverings of the same object, then $\{U_i \times_U V_j \longrightarrow U\}$ also is a covering. In fact, this notion is weaker then the one of a topology. This notion is what Groethendieck calls *pretopology.* There is an equivalence relation between pretopologies which is described in [Vi2] (pg.29, 33-39). The sheaf theory depends only on the equivalence class of pretopologies.

In what follows, a set $\{U_i \longrightarrow U\}$ of functions is called *jointly surjective* when the set-theoretic union of their images equals U.

Example 2.3. Obviously, the set of open coverings in $\mathcal{TOP}_X$ and in $\mathcal{TOP}$ gives a Grothendieck topology for this two categories (here we just see that the new definition includes the usual definition of topology).

Remark 2.4. Here we give a little review of base change of morphisms, and of properties used in the following like flat, smooth, étale. Standard reference is [H], together with [AM] and [Mi].

Definition 2.5. For a property of a morphism, to be *invariant under base change*, means that for every $f : X \longrightarrow Y$ having that property, and every $g : Z \longrightarrow Y$, the induced morphism from the fiber product $X \times_Y Z$ to Z has the same property.

Definition 2.6. A morphism $f : X \longrightarrow Y$ of schemes is *flat* if for all $x \in X$, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-algebra.

Typical examples of flat morphisms are open embeddings and projections from a product onto one factor.

Definition 2.7 ([Fu] B.2.5)**.** A morphism $f : X \longrightarrow Y$ has *relative dimension* n if for all subvarieties V of Y, and all irreducible components V' of $f^{-1}(V)$, $\dim(V') = \dim(V) + n$.

If f is flat, Y is irreducible, and X has pure dimension equal to $\dim(Y)+n$, then f has relative dimension n, and all base extensions $X \times_Y Y' \longrightarrow Y'$ have relative dimension n [[H] 3.8.6].

If $f : X \longrightarrow Y$ is a morphism, the sheaf of *relative differentials* is denoted $\Omega^1_{X|Y}$. If $g : Y \longrightarrow S$ is a morphism, there is an exact sequence of sheaves on X:

$$f^*\Omega^1_{Y|S} \longrightarrow \Omega^1_{X|S} \longrightarrow \Omega^1_{X|Y} \longrightarrow 0$$

[cfr. [H] 2.8.11].

Definition 2.8. A morphism $f : X \longrightarrow Y$ is *smooth* if f is flat of some relative dimension n, and $\Omega^1_{X|Y}$ is a locally free sheaf of rank n.

It follows that for any $Y \longrightarrow Y'$, the base change is also smooth of relative dimension n.

A morphism is *smooth of relative dimension* n if it is flat and $\Omega_{X|Y}$ is locally free of rank n.

Definition 2.9. A morphism $f : X \longrightarrow Y$ is *étale* if it is smooth of relative dimension 0. It is *unramified* if for all $x \in X$, $y = f(x)$, $m_y \mathcal{O}_X = m_x$.

Proposition 2.10 ([H] ex. 3.10.3)**.** *The following are equivalent:*

- *f is étale;*
- *f is flat, and $\Omega_{X|Y} = 0$;*
- *f is flat and unramified.*

Remark 2.11. Recall that if X and Y are smooth, then f is smooth iff it is submersive, f is étale iff $df(x)$ is an isomorphism for all $x \in X$. We will also need later that flat, smooth and étale are invariant under base change.

Proposition 2.12 ([Mi] prop. 2.11, prop. 2.12)**.** *Let $\phi : X \longrightarrow Y$ be an étale morphism, then:*

1. *For all $x \in X$, $\mathcal{O}_{x,X}$ and $\mathcal{O}_{\phi(x),Y}$ have the same Krull dimension.*
2. *The morphism ϕ is quasifinite.*
3. *The morphism ϕ is open.*
4. *If Y is respectively normal, regular, reduced, then the same property holds for X.*

Furthermore, the following properties hold:

- *Any open embedding is étale.*
- *The composite of two étale morphisms is étale.*

- *If $\phi \circ \psi$ and ϕ are étale then ψ is étale too.*

Example 2.13. (The global étale topology) This is a Grothendieck topology on $\mathcal{SCH}_{\mathbb{K}}$. A covering $\{U_i \longrightarrow U\}$ is a jointly surjective collection of étale morphisms. In a similar way there is a smooth and also a flat Grothendieck topology on $\mathcal{SCH}_{\mathbb{K}}$.

Definition 2.14. A contravariant functor: $\mathcal{C} \longrightarrow \mathcal{SET}$ is a sheaf for a given Grothendieck topology on $\mathcal{C}$ when the following diagram is exact:

$$F(U) \longrightarrow \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

for every U object in $\mathcal{C}$ and for every covering $(U_\alpha \longrightarrow U)$ of U.

We will see soon that stacks are a generalization of sheaves of sets. In order to have that stacks are somehow an extension of the given category, we need to recover all the original objects as sheaves of sets, i.e. we need the Yoneda functor to be a sheaf.

Definition 2.15. A Grothendieck topology $\mathcal{T}$ on a category $\mathcal{C}$ is called *subcanonical* if every representable functor is a sheaf with respect to $\mathcal{T}$.

Remark 2.16. The natural topologies on the categories of Topological spaces, Manifolds, schemes, Vector Bundles, Principal bundles are all subcanonical. This is described in [Vi2] (pg.40), we will work out schemes. To get an example of a non subcanonical site, take topological spaces, and as a Grothendieck topology, take the collection of all jointly surjective continuous morphisms. Let X the topological space with two points and the discrete topology and let Y be the two point space with the indiscrete topology. Now let $\{Y_1, Y_2\}$ be the covering of Y with its two points. There are two noncostant (hence noncontinuous) functions from Y to X, each belonging to $h_X(Y_1) \coprod h_X(Y_2)$. Conditions on double intersections are trivially satisfied since there are no double intersections, but these functions clearly don't glue to form a continuous function in $h_X(Y)$.

Theorem 2.17. *Let Y be a scheme. Then h_Y is a sheaf for the Zariski topology.*

Proof. Let X be a scheme and $\{U_i\}$ be an open covering, where each U_i has the subscheme structure. We will call arrows in the following way:

$$\begin{tikzcd} U_{ij} \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] & U_i \arrow[d, "p_i"] \\ U_j \arrow[r, "p_j"] & X \end{tikzcd}$$

where all arrows are open embedding. We have to show the exactness of the following diagram:

$$h_Y(X) \longrightarrow \prod_i h_Y(U_i) \rightrightarrows \prod_{i,j} h_Y(U_{ij})$$

where arrows are obtained applying h_Y to p_i and π_i. Since morphisms of schemes can be defined locally and then glued, we have to wonder only about unicity. Let's suppose there are two different morphisms α and $\bar{\alpha}$ between X and Y such that, once restricted to the U_i's are equal. As maps between the underlying topological spaces they are equal. Let's look at the induced maps on structure sheaves:

$$\alpha^\sharp : \mathcal{O}_Y \longrightarrow \alpha_* \mathcal{O}_X$$

$$\bar{\alpha}^\sharp : \mathcal{O}_Y \longrightarrow \alpha_* \mathcal{O}_X$$

Let $f \in \mathcal{O}_Y(U)$, then for all i, $\alpha^\sharp(f)$ and $\bar{\alpha}^\sharp(f)$ are equal once restricted to $\alpha^{-1}(U) \cap U_i$. Since $\alpha_*(\mathcal{O}_X)$ is a sheaf, $\alpha^\sharp(f) = \bar{\alpha}^\sharp(f)$. This proves unicity, hence exactness. □

The following theorem is due to Grothendieck:

Theorem 2.18 (cf. [Vi2] pg.40)**.** *Let Y be a scheme. Then h_Y is a sheaf in the étale topology.*

3 Stacks

A stack is a sheaf of groupoids. In the following definition, to simplify notation we denote f^* (as usual) for $F(f)$ (F a pseudofunctor and f a morphism in $\mathcal{C}$).

Definition 3.1. Let $F : \mathcal{C} \longrightarrow \mathcal{GPD}$ be a pseudofunctor in groupoids from a category with a Grothendieck topology, C be an object of the category and $\{U_i \xrightarrow{f_i} U\}$ be a covering. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a collection of objects

$\xi_i \in F(U_i)$, together with isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\xi_j \simeq \mathrm{pr}_1^*\xi_i$ in $F(U_i \times_U U_j)$, such that the following cocycle condition is satisfied.

For any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^*\phi_{ik} = \mathrm{pr}_{12}^*\phi_{ij} \circ \mathrm{pr}_{23}^*\phi_{jk} : \mathrm{pr}_3^*\xi_k \longrightarrow \mathrm{pr}_1^*\xi_i$$

where pr_{ab} and pr_a are projections on the a^{th} and b^{th} factor, or the a^{th} factor respectively.

The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.

An arrow between objects with descent data

$$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \longrightarrow (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i : \xi_i \longrightarrow \eta_i$ in $F(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{tikzcd}
\mathrm{pr}_2^*\xi_j \arrow[r, "\mathrm{pr}_2^*\alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^*\eta_j \arrow[d, "\psi_{ij}"] \\
\mathrm{pr}_1^*\xi_i \arrow[r, "\mathrm{pr}_1^*\alpha_i"] & \mathrm{pr}_1^*\eta_i
\end{tikzcd}$$

commutes.

In understanding the definition above it may be useful to contemplate the following cartesian cube (every face is cartesian):

$$\begin{tikzcd}
& U_{ijk} \arrow[rr, "\mathrm{pr}_{23}"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dd, "\mathrm{pr}_{13}"] & & U_{jk} \arrow[dl] \arrow[dd] \\
U_{ij} \arrow[rr] \arrow[dd] & & U_j \arrow[dd] & \\
& U_{ik} \arrow[rr] \arrow[dl] & & U_k \arrow[dl] \\
U_i \arrow[rr] & & U &
\end{tikzcd}$$

in which all arrows are given by projections.

There is an obvious way of composing morphisms, which makes objects with descent data the objects of a category, denoted by $F(\mathcal{U}) = F(\{U_i \longrightarrow U\})$. For each object ξ of $F(U)$ we can construct an object with descent data on a covering $\{\sigma_i : U_i \longrightarrow U\}$ as follows. The objects are the pullbacks $\sigma_i^*\xi$; the isomorphisms $\phi_{ij} : \mathrm{pr}_2^*\sigma_j^*\xi \simeq \mathrm{pr}_1^*\sigma_i^*\xi$ are the isomorphisms that come from the fact that both $\mathrm{pr}_2^*\sigma_j^*\xi$ and $\mathrm{pr}_1^*\sigma_i^*\xi$ are pullbacks of ξ to U_{ij}. If we identify $\mathrm{pr}_2^*\sigma_j^*\xi$ with $\mathrm{pr}_1^*\sigma_i^*\xi$, as it is commonly done, then the ϕ_{ij} are identities.

Given an arrow $\alpha : \xi \longrightarrow \eta$ in $F(U)$, we get arrows $\sigma_i^* : \sigma_i^*\xi \longrightarrow \sigma_i^*\eta$, yielding an arrow from the object with descent associated with ξ to the one associated with η. This defines a functor $\mathcal{A} : F(U) \longrightarrow F(\{U_i \longrightarrow U\})$.

Definition 3.2 ([Vi2] (pg.82))**.** Let $\mathcal{C}$ be a category with a Grothendieck topology. A pseudofunctor (presheaf), is said respectively:

1. *separated prestack* if the functor $\mathcal{A}$ is fully faithful.

2. *stack* if the functor $\mathcal{A}$ is an equivalence of categories.

A stack is a sheaf of groupoids, i.e. a pseudofunctor in groupoids (presheaf) which satisfies the sheaf axioms.

Remark 3.3. Letus give now a definition a little less precise but a little more concise. The main problem here is that $(U_{i,j} \times_{U_i} U_{i,k})$ is not equal to $(U_{i,k} \times_{U_k} U_{j,k})$: they are only canonically isomorphic. This is taken from [G]:

Let $\{U_i \xrightarrow{f_i} U\}$ be a covering of U. Then (we write down the two conditions in a simpler way):

1. (glueing of (iso)morphisms and unicity-the functor $\mathcal{A}$ is fully faithful) if X and Y are two objects of $F(U)$, and $\phi_i : f_i^*(X) \longrightarrow f_i^*(Y)$ are (iso)morphisms such that $f_{ij}^*(\phi_i) = f_{ij}^*(\phi_j)$, then there exists a unique morphism $\eta : X \longrightarrow Y$ such that $f_i^*(\eta) = \phi_i$. This exactly means that the functor:

$$\mathrm{Iso}_U(X, Y) : \mathcal{C} \longrightarrow \mathcal{SET}$$

which associates to a morphism $f : U_i \longrightarrow U$ the set of isomorphisms in $F(U_i)$ between f^*X and f^*Y is a sheaf for the given topology.

2. (glueing of objects: the functor $\mathcal{A}$ is essentially surjective, every descent datum is effective) If X_i are objects of $F(U_i)$, and $\phi_{ij} : f_{ij}^*(X_j) \longrightarrow f_{ij}^*(X_i)$ are morphisms satisfying the cocycle condition $f_{ijk}^*(\phi_{ij}) \circ f_{ijk}^*(\phi_{jk}) = f_{ijk}^*(\phi_{ik})$, then there exists an object $X \in F(U)$ and $\phi_i : f_i^*(X) \xrightarrow{\cong} X_i$ such that $\phi_{ji} \circ f_{ij}^*(\phi_i) = f_{ij}^*(\phi_j)$.

Remark 3.4. Now, in order to have a notation which extends the existing one, we need to check the following: if the pseudofunctor in groupoids factors via $\mathcal{SET}$, then it is a stack if and only if it is a sheaf of sets. It is a separated presheaf if and only if it is a separated prestack. [[LM], 3.4.1]

Proof. First, let F be a stack. Let's consider an object X and a covering $\{X_i \xrightarrow{p_i} X\}$ in the given Grothendieck topology. We have to check that the following diagram is exact:

$$F(X) \longrightarrow \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

let $\{\xi_i\}$ be an element of $\prod F(X_i)$ such that $\xi_i|_{X_i \times_X X_j} = \xi_j|_{X_i \times_X X_j}$. Equality is trivially an isomorphism satisfying the cocycle condition, so there exists (thanks to the second point in the definition of stack) $\xi \in F(X)$ and isomorphisms $\psi_i : \xi|_{X_i} \longrightarrow \xi_i$ in $F(X_i)$. F factors via $\mathcal{SET}$, so isomorphisms ψ_i are identities, and so $\xi|_{X_i} = \xi_i$ for all i. Let now $\bar{\xi}$ be one other element in $F(X)$ with the same property as ξ. So $\bar{\xi}|_{X_i}$ and $\xi|_{X_i}$ are equal, and so isomorphic with the identity (which clearly satisfies the conditions in 1). So by the first point, there exists an isomorphism globally between $\bar{\xi}$ and ξ. It is a morphism between sets, and locally it is the identity, therefore it is the identity globally and $\bar{\xi} = \xi$. Now let's see that a sheaf is a stack. Conversely, let F be a sheaf. Isomorphisms are a sheaf just because the pseudofunctor factorizes through $\mathcal{SET}$: the sheaf condition is not needed here. It is possible to define a global isomorphism just defining it on any single element of the set thanks to the local isomorphisms. The condition on the intersections tells that the global map defined is well defined. The uniqueness is clear (the global map has to be equal to the local maps once restricted). Every descent datum is effective. In fact, fix a descent datum $X_i, \phi_{i,j}$. The equality condition on the triple intersections tells that the isomorphisms on the double intersections are actually equalities. But the sheaf condition for a contravariant functor in $\mathcal{SET}$ tells now what is needed: equality on the double intersection is the right condition to glue. □

Definition 3.5. A *morphism of stacks* is simply a morphism of pseudofunctors in groupoids. The same holds for isomorphisms.

Remark 3.6. If the two pseudofunctors factorize via $\mathcal{SET}$, then f is a morphism of stacks if and only if it is a morphism of sheaves.

Example 3.7. ($Vect_n$) Here the category $\mathcal{C}$ could be the category of topological spaces, of schemes, of manifolds,... Let X be an object of the category. $Vect_n(X)$ is the groupoid of rank-n vector bundles over the base X. Let $f : X' \longrightarrow X$ be a morphism. $Vect_n(f)$ is the functor pull-back of vector spaces, i.e. if $\pi : E \longrightarrow X$ is a vector bundle, $Vect_n(f)$ is what is usually called f^*:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

Let's check it is a stack. This turns out to be quite trivial:

1. If E and E' are vector bundles over X, and $\{U_i\}$ is a covering of X, then a set of (iso)morphisms $f_i : E \restriction_{U_i} \longrightarrow E' \restriction_{U_i}$ which are equal on the double intersections glues to an isomorphism between E and E'

2. By definition, a vector bundle is exactly a local trivial bundle with isomorphisms satisfying the cocycle condition.

For the following example, recall the (here Grp is the category of groups):

Definition 3.8. A *group object* of $\mathcal{C}$ is an object G of $\mathcal{C}$, together with a functor $\mathcal{C}^{op} \longrightarrow Grp$, whose composite with the forgetful functor in $\mathcal{SET}$ equals h_G.

To get the usual form of a group object inside a category, one has just to apply Yoneda's lemma. This is worked out in [[Vi2] prop. 2.12]. For schemes, this is the same given in [[H] pg.324]. For varieties, this is the same given in [[H] I, Ex. 3.21].

Example 3.9. BG This is also known in the literature as *trivial gerbe.* Take the same category as in the previous example. With the same notations, $BG(X)$ is the groupoid of principal G-bundles over X, and $BG(f)$ is the pull-back morphism for principal bundles. To prove that this is a stack is exactly the same as in the previous example. If G is taken to be $GL(n)$, we can show that BG and $Vect_n$ are isomorphic as stacks. There are two morphisms of pseudofunctors: $f : BGL(n) \longrightarrow Vect_n$ and $g : Vect_n \longrightarrow BGL(n)$. Let's describe them and see they give an isomorphism of pseudofunctors. $f(P \longrightarrow M)$ by definition is $P \times V/\cong$ where $(p, v) \cong (p', v')$ whenever exists $g \in G$ and $p = gp$, $v = g(v)$. This is a vector bundle with the same transition functions of the principal bundle associated. Conversely, if $E \longrightarrow M$ is a vector bundle, we build-up a principal $GL(n)$-bundle as

$$P := \coprod_\alpha U_\alpha \times G/\sim$$

with

$$(x_\alpha, g_\alpha) \sim (x_\beta, g_\beta) \Longleftrightarrow x_\alpha = x_\beta \in U_\alpha \cap U_\beta \text{ and } g_\beta = \phi_{\beta\alpha}(x)g_\alpha$$

where (U_α) is a local trivialization of E with transition functions $\phi_{\beta\alpha}$. We now let $g(E \longrightarrow M) = (P \longrightarrow M)$. This gives an isomorphism between the two pseudofunctors. In a similar way, for a given group G, a representation ρ in $Aut(V)$ gives a morphism $BG \longrightarrow Vect_n$, where $n = \dim(V)$.

If H is a normal subgroup of G and $\Gamma = G/H$, there is a morphism of pseudofunctors $BG \longrightarrow B\Gamma$ which sends a principal $G-$ bundle $E \longrightarrow X$ to the principal Γ-bundle $E/H \longrightarrow X$. If C is an object of the category, there is a morphism of pseudofunctors $C \longrightarrow BG$, sending $X \longrightarrow C$ to the trivial G-bundle $X \times_C G \longrightarrow X$.

Example 3.10. Quotient stacks. This example generalizes the previous one. As usual X is an object of $\mathcal{C}$ and G is a group object of the category, acting on the right on X. We define the stack $[X/G]$. On objects, $[X/G](Y)$ is, by definition the groupoid given by:

$$\begin{tikzcd} E \arrow[r, "\alpha"'] \arrow[d, "\pi"] & X \\ Y \end{tikzcd}$$

where π gives a principal $G-$bundle, and α is a $G-$equivariant morphism. This datum will be written down in the more compact form: (Y, π, E, α). If $f : Y \longrightarrow Y'$ is a morphism, $[X/G](f)$ is the homomorphism of groupoids given by what is usually called f^*:

$$\begin{tikzcd} f^*(E') \arrow[rr] \arrow[dr] \arrow[dd] & & E' \arrow[dl] \arrow[dd] \\ & X & \\ Y \arrow[rr] & & Y' \end{tikzcd}$$

This pseudofunctor gives rise to a stack. If X turns out to be the terminal object of the category, this example reduces to the previous one.

Definition 3.11. A diagram of morphism of stacks:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr, "h"'] & G \arrow[d, "g"] \\ & H \end{tikzcd}$$

2-commutes if it is given an isomorphism of functors $h \cong g \circ f$.

Example 3.12. The morphism $C \longrightarrow BG$ sending $X \longrightarrow C$ to the trivial G-bundle $X \times_C G \longrightarrow X$

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & BG \arrow[d] \\ & B\Gamma \end{tikzcd}$$

where f, g, h were defined while BG was defined. The diagram 2-commutes, because $X \times_C \Gamma \longrightarrow X$ is canonically isomorphic to $(X \times_C F)/H$ but h is not equal to $g \circ f$.

Definition 3.13. Let $f_1 : F_1 \longrightarrow G$, $f_2 : F_2 \longrightarrow G$ be morphisms of stacks. The *fiber product* $F_1 \times_G F_2$ is the stack defined as follows: on a scheme X its objects are triples (ξ_1, ξ_2, α), where ξ_i is an object of $F_i(X)$ and $\alpha : f_1(\xi_1) \longrightarrow f_2(\xi_2)$ is an isomorphism. Arrows from (ξ_1, ξ_2, α) to (η_1, η_2, β) are pairs (ϕ_1, ϕ_2), $\phi_i : \xi_i \longrightarrow \eta_i$ an arrow in $F_i(X)$, and $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$.

Remark 3.14. A 2-commutative diagram of stacks,

$$\begin{tikzcd} H \arrow[r] \arrow[d] & F_2 \arrow[d] \\ F_1 \arrow[r] & G \end{tikzcd}$$

induces a morphism $g : H \longrightarrow F_1 \times_G F_2$, unique up to canonical isomorphism. If g is an isomorphism, we say that the diagram is 2-cartesian, and we also call H the fiber product.

Remark 3.15. If $F, G, H : \mathcal{C} \longrightarrow \mathcal{SET}$ are sheaves on a category with a Grothendieck topology, then for all $\alpha : F \longrightarrow G$, $\beta : H \longrightarrow G$ morphisms, $F \times_G H$ is again a sheaf.

Lemma 3.16. *[Vi2] Let $F, G, H : \mathcal{C} \longrightarrow \mathcal{GPD}$ be stacks. Then for all $\alpha : F \longrightarrow G$, $\beta : H \longrightarrow G$, $F \times_G H$ is again a stack.*

Remark 3.17. Let U be an object of the category $\mathcal{C}$. Consider the diagonal $\Delta_U : U \longrightarrow U \times U$. If V is another object and a morphism $V \longrightarrow U \times U$ is given, this corresponds to two objects X and Y in $h_U(V)$. Then the fiber product $U \times_{U \times U} V$ is equivalent to the sheaf $\mathrm{Iso}_V(X, Y)$. In fact, by P the fiber product, one has for any object B:

$$P(B) = \{(f_1, f_2, \alpha)\ f_1 \in h_U(B), f_2 \in h_V(B),\ \alpha : (f_2, f_2) \longrightarrow (x_1 \circ f_1, x_2 \circ f_1)\}$$

and this is equivalent to:

$$\mathrm{Iso}_U(x_1, x_2)(B) = \{g \in h_U(B), \beta : g^*x_1 \longrightarrow g^*x_2\}$$

Definition 3.18. Let $f : F \longrightarrow G$ be a morphism of stacks. It is said to be *representable* if for any object $Y \in \mathcal{C}$ and any morphism $Y \longrightarrow G$, the fiber product $F \times_G Y$ is representable.

Proposition 3.19 ([Vi1] prop. 7.13). *Let F be a stack. Δ_F is representable if and only if for every C object of $\mathcal{C}$, $h_C \longrightarrow F$ is representable.*

Proof. Assume that Δ_F is representable. If $f : X \longrightarrow F$ and $g : Y \longrightarrow F$ are morphisms where X and Y are objects of the category $\mathcal{C}$, we have a cartesian diagram:

$$\begin{tikzcd} h_X \times_F h_Y \arrow[r] \arrow[d] & h_X \times h_Y \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times F \end{tikzcd}$$

Hence $h_X \times_F h_Y$ is representable. Suppose that every morphism from a scheme to F is representable. Let $f : h_X \longrightarrow F \times F$ be a morphism, corresponding o two morphisms $f_1 : X \longrightarrow F, f_2 : X \longrightarrow F$. Then $f = (f_1 \times f_2) \circ \Delta_X$. The following diagram is cartesian:

$$\begin{tikzcd} F \times_{F\times F} h_X \arrow[r] \arrow[d] & h_X \arrow[d] \\ h_X \times_F h_X \arrow[r] \arrow[d] & h_X \times h_X \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times F \end{tikzcd}$$

Then $h_X \times_F h_X$ is a scheme, and therefore also $F \times_{F\times F} h_X = (h_X \times_F h_X) \times_{X\times X} X$ is representable. $\square$

Definition 3.20. If P is a property which is stable under base change, we say that a representable morphism $f : F \longrightarrow G$ of stacks has P if for any object in the category and every stack morphism $X \longrightarrow G$, the induced morphism $X \times_G F \longrightarrow X$ has P.

What follows (Grothendieck representability theorem) could be used to give a new definition of scheme, given in this setting:

Theorem 3.21. *A sheaf $F : \mathcal{SCH}_{\mathbb{K}} \longrightarrow \mathcal{SET}$ is representable if and only if there exist F_i representable functors and $\phi_i : F_i \longrightarrow F$ representable morphisms such that:*

1. *F_i is represented by an affine scheme.*

2. *ϕ_i is an open embedding.*

3. *$\coprod F_i \longrightarrow F$ is a jointly surjective covering, i.e. for all S scheme, for all $\alpha \in F(S)$, S_i defined by $h_{S_i} \cong F_i \times_F h_S$, $\coprod f_i(S_i) = S$.*

Proof. Without loss of generality, let's assume $F_i = h_{X_i}$ for X_i scheme (without such assumption notation becomes a little more complicated).

$$\begin{tikzcd} F_{ij} := F_i \times_F F_j \arrow[r, "p_i"] \arrow[d, "p_j"'] & F_i \arrow[d, "\alpha_i"] \\ F_j \arrow[r, "\alpha_j"] & F \end{tikzcd}$$

The p_i's correspond to open embeddings of schemes $U_i \cap U_j \longrightarrow U_i$. Let's call $U_{ij} := U_i \cap U_j$. Clearly $U_{ij} = U_{ji}$, hence we can glue the schemes U_i along U_{ij}'s to construct a new scheme X, which corresponds to the functor h_X. We now show that h_X and F are isomorphic. We have injective natural transformations : $\alpha_i : F_i \longrightarrow F$. $\alpha_i \upharpoonright_{U_{ij}}: h_{U_i} \upharpoonright_{U_{ij}} \longrightarrow F$ is equal to $\alpha_j \upharpoonright_{U_{ij}}: h_{U_j} \upharpoonright_{U_{ij}} \longrightarrow F$. By Yoneda's Lemma, $\{\alpha_i\}$ corresponds to $\{b_i\} \in \prod_F(U_i)$ such that $b_i \upharpoonright_{F(U_{ij})} = b_j \upharpoonright_{F(U_{ij})}$. F is a sheaf, hence there exists a unique $b \in F(X)$ such that $b \upharpoonright_{F(U_i)} = b_i$ for all i, i.e. is defined the natural transformation $\beta : h_X \longrightarrow F$ glueing the isomorphisms β_i with $h_{U_{ij}}$ along intersections. For each scheme T, $\beta_i(T)$ and $\beta_{ij}(T)$ are isomorphisms, hence also $\beta(T)$ is an isomorphism. $\square$

This proposition tells us that among sheaves of sets (spaces), schemes are exactly the one with a Zariski open affine covering. One doesn't enlarge the category of schemes by gluing, if the Zariski topology is used. In fact, if one uses the étale topology, then a similar construction leads to the notion of an algebraic space.

Theorem 3.22. *The same proposition holds for Algebraic Spaces with the étale topology instead of the Zariski topology. [LM].*

As a standard reference for algebraic spaces we take [Kt].

4 Algebraic Stacks

This section is taken essentially from [G] and [Vi1]. Here the category $\mathcal{C}$ will be Sch/S.

$$\begin{tikzcd}
 & \mathit{Algebraic\ Stacks} \arrow[r] & \mathit{Stacks} \arrow[r] & \mathit{Presheaves\ of\ groupoids} \\
\mathit{Sch/S} \arrow[r] \arrow[ur] & \mathit{Algebraic\ spaces} \arrow[r] \arrow[u] & \mathit{Spaces} \arrow[r] \arrow[u] & \mathit{Presheaves\ of\ sets} \arrow[u]
\end{tikzcd}$$

Where every arrow means that the left hand side is a full subcategory (or full sub 2-category) of the right hand side.

Definition 4.1. A *space* is just a sheaf of sets. The remark 3.4 tells exactly that spaces are stacks factorizing via $\mathcal{SET}$.

Definition 4.2. An *algebraic space* is a space S such that:

1. the diagonal Δ_S is quasicompact and separated;

2. there is a scheme U and an étale surjective morphism $h_U \longrightarrow S$.

Definition 4.3. An *algebraic stack*, or *Deligne-Mumford stack* is a stack F such that:

1. the diagonal Δ_F is representable, quasicompact and separated;

2. there is a scheme U and an étale surjective morphism $h_U \longrightarrow F$.

such a surjective morphism $h_S \longrightarrow F$ is called an *étale atlas* for F.

These are the official definitions of algebraic spaces and algebraic stacks, in this work we will always start from the category $\mathcal{SCH}_{\mathbb{K}}$ of schemes of finite type over an algebraic closed field $\mathbb{K}$, hence the hypothesis of quasicompactness of the diagonal will be always satisfied.

Definition 4.4. A *morphism of algebraic stacks* is simply a morphism of stacks, hence a morphism between the pseudofunctors.

The following proposition gives an simpler criterium to determine whether or not an algebraic stacks has a given property which is local in the base.

Proposition 4.5. *Let $\phi : F \longrightarrow G$ be a representable morphism of algebraic stacks, and let $h_X \longrightarrow G$ be an étale atlas, P a property which is local in the base (in particular invariant under base change). Then the following are equivalent:*

1. *ϕ has the property P;*

2. *$h_X \times_G F$ has the property P.*

Proof. We have only to prove that (2) $\Rightarrow$ (1). Following the definition of having a property P for a representable morphism, we take a scheme A and a morphism $h_A \longrightarrow G$. Since $h_X \longrightarrow G$ is representable, we can define $h_B := h_A \times_G h_X$, so we have the following cartesian diagram:

$$\begin{tikzcd}
h_B = h_A \times_G h_X \arrow[r] \arrow[d, "\tilde{f}"'] & h_X \arrow[d] \\
h_A \arrow[r] & G
\end{tikzcd}$$

$h_X \longrightarrow G$ is surjective and étale, so the morphism $h_B \longrightarrow h_A$ is étale too. Now we have the following diagram, whose big and right squares are cartesian by definition:

$$\begin{tikzcd}
h_B \times_G F \arrow[r] \arrow[d, "\tilde{f}"'] & h_X \times_G F \arrow[r] \arrow[d, "f"'] & F \arrow[d] \\
h_B \arrow[r] & h_X \arrow[r] & G
\end{tikzcd}$$

It follows that the left square too is cartesian. $\tilde{f}$ has the property P, since f is invariant under base change. It follows that the left square is cartesian, and we have proven the statement. Now the following cartesian diagram, proves that g has P:

$$\begin{tikzcd}
h_B \times_G F \arrow[r] \arrow[d, "\tilde{f}"'] & h_A \times_G F \arrow[r] \arrow[d, "g"'] & F \arrow[d] \\
h_B \arrow[r] & h_A \arrow[r] & G
\end{tikzcd}$$

g has P because $\tilde{f}$ has P and $h_B \longrightarrow h_A$ is étale and surjective.

□

The following is taken from [Vi1]:

Proposition 4.6. *The diagonal of an algebraic stack is unramified.*

Proof. Let F be an algebraic stack with atlas U. Then $U \times U \longrightarrow F \times F$ is an atlas for $F \times F$. If Y is a scheme and $Y \longrightarrow F \times F$ a morphism, set $X = F \times_{F \times F} Y$. The following diagram is commutative:

$$\begin{tikzcd}
X \times_F U \arrow[rrr] \arrow[dr] \arrow[ddd] & & & Y \times_{F \times F} (U \times U) \arrow[dl] \arrow[ddd] \\
& U \arrow[r, "\Delta_U"] \arrow[d] & U \times U \arrow[d] & \\
& F \arrow[r, "\Delta_F"] & F \times F & \\
X \arrow[ur] \arrow[rrr] & & & Y \arrow[ul]
\end{tikzcd}$$

Of the four external squares, all are cartesian by definition, except for the upper one. It follows that the top square is also cartesian. Hence the top row is an embedding, the left and right columns are étale, and therefore $X \longrightarrow Y$ is unramified. □

Also a sort of converse holds, but it is considerably more difficult.

Theorem 4.7 ([LM] Thm 8.1, [DM] Thm 4.21)**.** *Let F be a stack such that:*

1. *The diagonal Δ_F is representable.*
2. *There is a scheme U and a smooth surjective morphism $h_U \longrightarrow F$.*

Then F is an algebraic stack of Deligne Mumford if and only if the diagonal Δ_F is unramified.

Remark 4.8. If $\xi \in F(X)$, the sheaf of groups $\mathrm{Iso}_X(\xi, \xi)$ is a separated group scheme on X, which is unramified. $\mathrm{Aut}_X(\xi)$ can have only finitely many sections, and therefore ξ has only finitely many automorphisms in $F(X)$. The presence of nontrivial automorphisms in some $F(X)$ implies that the diagonal $F \longrightarrow F \times F$ is not an embedding, i.e. the algebraic stack turns out not to be a scheme.

Definition 4.9 ([Vi1])**.** [*separated morphism*] A morphism of stacks $f : F \longrightarrow G$ is *separated* if for any complete discrete valuation ring R and any commutative diagram:

$$\begin{tikzcd} & F \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & G \end{tikzcd}$$

any isomorphism between the restrictions of g_1 and g_2 to the generic point of $\mathrm{Spec}(R)$ can be extended to an isomorphism between g_1 and g_2.

Definition 4.10 ([Vi1])**.** [*proper morphism*] A morphism of stacks $f : F \longrightarrow G$ is *proper* if it is separated, and for any complete valuation ring R with field of fractions K and any commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "g"] \arrow[d] & F \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & G \end{tikzcd}$$

there exists a finite extension K' of K such that the morphism $\mathrm{Spec}(K') \longrightarrow F$ induced by g extends to $\mathrm{Spec}(R')$, where R' is the integral closure of R in K'.

Lemma 4.11 ([Vi1])**.** *Composite of separated, composite of proper, composite of representable is still separated, proper, representable. A morphism of stacks $F \longrightarrow G$ is separated if and only if the diagonal $F \longrightarrow F \times_G F$ is proper.*

Lemma 4.12 ([Vi1])**.** *Separated, proper and representable are invariant under base change.*

Lemma 4.13 ([Vi1])**.** *If a morphism of stacks is representable, then it is separated or proper if and only if it is represented by separated or proper morphisms of schemes*

Remark 4.14. Our definition of separated and proper morphisms works for all morphisms of stacks, even if they are not representable.

Definition 4.15. A *substack* of a stack F is a representable morphism of stacks $F' \longrightarrow F$ which is represented by embeddings of schemes. A substack is *open or closed* whether the representing embeddings of schemes are open or closed.

Definition 4.16. A stack is called *connected* if it is not the disjoint union of two proper open substacks. It is called *irreducible* if it is not the union of two proper closed substacks. Is called *integral* if it is both reduced and irreducible.

Recall that étale morphisms preserve dimension.

Definition 4.17. The *dimension* of an algebraic stack is just the dimension of an étale atlas.

5 Global quotients

In the following, we work out the notion of algebraic stack for the example of a global quotient $[X/G]$. The description of the pseudofunctor (hence stack) $[X/G]$ was given in a previous remark. Here we look at its algebraic structure, proving it is an algebraic Deligne Mumford stack. We want to show that an étale atlas is triv: $X \longrightarrow [X/G]$, where the morphism triv is given, via 2-Yoneda's Lemma, by the object of $[X/G]$:

$$\begin{tikzcd} X \times G \arrow[r, "\mathrm{act}"] \arrow[d, "pr_1"'] & X \\ X & \end{tikzcd}$$

where act is the action of the algebraic group onto X and pr_1 is the projection onto the first factor. We need to prove at first that this morphism is representable. We denote, using a standard convention, by $\bar{f}$ the lifting morphism of f via a fixed pullback.

Lemma 5.1. *Let B be a scheme. A morphism $B \longrightarrow [X/G]$ is given (via 2-Yoneda's lemma) by an object of $[X/G](B)$, i.e. a principal $G-$bundle E with a $G-$ equivariant morphism to X:*

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "\pi"'] & X \\ B & \end{tikzcd}$$

then the following diagram is 2-cartesian:

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "\pi"'] & X \arrow[d, "triv"] \\ B \arrow[r, "b"] & {[X/G]} \end{tikzcd}$$

Proof. We will show that the stack E satisfies the universal property of the fiber product. First we will prove that the diagram of the lemma is 2-commutative. We will only check it on the objects of $E(Z)$. Let $f : Z \longrightarrow E$ be an object of $E(Z)$ This object is sent by:

$$b \circ \pi(f) := (E \times_B Z, Z, a \circ \overline{\pi \circ f})$$

$$\text{triv} \circ a(f) := (Z \times G, Z, act \circ (a \circ f, Id))$$

To show that this diagram is 2-commutative, we have to show that there exists an isomorphism, denoted $\alpha(f)$, between these two objects that is:

- $\alpha(f)$ is an isomorphism of $G-$bundle over Z;
- the equality $a \circ \overline{\pi \circ f} \circ \alpha(f) = act \circ (a \circ f, Id)$ holds.

Let's collect all these maps in a unique diagram:

$$\begin{tikzcd}
 & & Z \times_B E \arrow[d] \arrow[rrd, "\overline{\pi \circ f}"] & & & \\
Z \times G \arrow[rru, dotted, "\alpha(f)"] \arrow[rr] \arrow[rrdd] & & Z \arrow[rd, "f"] & & E \arrow[r, "a"] \arrow[d, "\pi"'] & X \\
 & & & E \arrow[r, "\pi"] \arrow[d, "b"] & B & \\
 & & X \times G \arrow[r, "pr_1"] \arrow[d, "act"'] & X & & \\
 & & X & & &
\end{tikzcd}$$

In order to define $\alpha(f)$, we can use the universal property of $Z \times_B E$, namely there exists a unique morphism, denoted $\alpha(f)$ such that the following diagram is commutative:

$$\begin{tikzcd}
Z \times G \arrow[rd] \arrow[rrd, bend left, "act_E \circ (f, Id)"] \arrow[rdd, bend right, "pr_1"'] & & \\
 & Z \times_B E \arrow[r, "\overline{\pi \circ f}"'] \arrow[d, "pr_1"] & E \arrow[d, "\pi"] \\
 & Z \arrow[r, "\pi \circ f"] & B
\end{tikzcd}$$

where act_E is the action of G onto E. $act_E \circ (f, Id)$ and $\pi \circ f$ are $G-$ equivariant. Hence $\alpha(f)$ is $G-$ equivariant, and so a morphism of G-bundles over Z that is an isomorphism. Let now show the equality $a \circ \overline{\pi \circ f} \circ \alpha(f) = act \circ (a \circ f, Id)$. The last diagram implies that $act_E \circ (f, Id) = \overline{\pi \circ f} \circ \alpha(f)$. We compose by the morphism a both sides and use that b is $G-$ equivariant. Then we get the equality $a \circ \pi \circ f \circ \alpha(f) = act \circ (a \circ f, Id)$, and so the diagram of our lemma is 2-commutative. Now we will prove that this diagram is cartesian. There will be some simplification since we will prove it only on objects. Let $\mathcal{Z}$ be a stack, and let's consider the following 2-commutative diagram:

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "triv"] \\
B \arrow[r, "b"] & {[X/G]}
\end{tikzcd}$$

This means that for any $z \in \mathcal{Z}(Z)$, there exists an isomorphism $\alpha(z)$ between the objects:

$$triv \circ g(z) = (Z \times G, Z, act \circ f(z))$$

$$b \circ f(z) = (Z \times_B E, Z, a \circ f(z))$$

Summing up all the datas in the following commutative diagram:

$$\begin{tikzcd}
 & & Z \arrow[d, "s"] & & & & \\
 & & Z \times G \arrow[rr, dotted, "\alpha(z)", "\equiv"'] \arrow[dl, "\overline{g(z)}"'] \arrow[dr] & & Z \times_B E \arrow[dl] \arrow[dr, "\overline{f(z)}"] & & \\
 & X \times G \arrow[dl, "act"'] \arrow[dr, "pr_1"] & & Z \arrow[dl, "g(z)"'] \arrow[dr, "f(z)"] & & E \arrow[dl, "\pi"'] \arrow[dr, "a"] & \\
X & & X & & B & & X
\end{tikzcd}$$

here $s : Z \longrightarrow Z \times G$ sends x to (x, e) turns out to be useful later. Here $\alpha(f)$ is chosen to satisfy the following equality:

$$act \circ \overline{g(z)} = a \circ \overline{f(z)} \circ \alpha(z)$$

Now we define the following morphism on objects:

$$\mathcal{H} : \mathcal{Z} \longrightarrow E$$

$$z \longrightarrow \overline{f(z)} \circ \alpha(z) \circ s : Z \longrightarrow E$$

$$(\phi : z \longrightarrow z') \longrightarrow (\mathcal{Z}(\phi) : Z \longrightarrow Z')$$

where $s : Z \longrightarrow Z \times G$ sends $t \longrightarrow (t, e)$. A look at the previous big commutative diagram let's understand that $f = \pi \circ \mathcal{H}$, while the equality $g = a \circ \mathcal{H}$ holds thanks to the previous equation (look always at the big commutative diagram). Then the stack E satisfies the universal property of the fiber product, hence the lemma. $\square$

Lemma 5.2. *Let $(B \times G, \pi, B, a)$ be an object in $[X/G]$. Then the stack morphism induced from B to $[X/G]$ via 2-Yoneda lemma factors through the stack morphism triv defined before.*

Proof. We consider the section $s : B \longrightarrow B \times G$ sending $x \longrightarrow (x, e)$. Then we get a scheme morphism $a \circ s : B \longrightarrow X$. Let's prove that the following diagram is 2-commutative:

$$\begin{tikzcd} & X \arrow[d, "triv"] \\ B \arrow[ur, "a \circ s"] \arrow[r, "b"'] & {[X/G]} \end{tikzcd}$$

Let $f : Z \longrightarrow B$ be a scheme morphism, i.e. an object of B. Then we have:

$$b(f) = (Z \times G, Z, a \circ (g, Id))$$

and

$$triv \circ a \circ s(f) = (Z \times G, Z, act \circ (a \circ s \circ f, Id))$$

We will prove the following equality:

$$act \circ (a \circ s \circ f, Id) = a \circ (f, Id)$$

We have that, as a is $G-$equivariant:

$$act \circ (a, Id) = a \circ act_{B \times G}$$

where

$$act_{B\times G} : (B \times G) \times G \longrightarrow B \times G$$

$$(x, \lambda, \mu) \longrightarrow (x, \lambda\mu)$$

The scheme morphism s defines a morphism $(s, Id) : B \times G \longrightarrow (B \times G) \times G$. We have that:

$$act_{B\times G} \circ (s, Id) = Id_{B\times G}$$

Then the equality stated before implies the equality desired. We deduce that the functors $triv \circ (a \circ s)$ and b are the same on the objects of B. In the same way, it is possible to show the equality on the morphisms of X. $\square$

Theorem 5.3. *Let G be a group scheme acting on X with finite and reduced stabilizers, where X is a separated scheme (of finite type over an algebraically closed field) then the stack $[X/G]$ is an algebraic stack.*

Proof. Using simultaneously last two lemmas, one gets the following cartesian diagram:

$$\begin{tikzcd} X \times G \arrow[r, "pr_1"] \arrow[d, "act"'] & X \arrow[d, "triv"] \\ X \arrow[r, "triv"] & {[X/G]} \end{tikzcd}$$

A standard argument in category theory guarantees that the following is cartesian:

$$\begin{tikzcd} X \times G \arrow[r, "{pr_1, act}"] \arrow[d] & X \times X \arrow[d] \\ {[X/G]} \arrow[r, "\Delta_{[X/G]}"] & {[X/G] \times [X/G]} \end{tikzcd}$$

here Δ as usual denotes the diagonal morphism. By definition, the morphism of stack $triv : X \longrightarrow [X/G]$ (which we have just seen is representable) is surjective and smooth iff the scheme morphism $\pi : E \longrightarrow B$ is surjective and smooth. As P is a $G-$bundle over B, the projection is smooth and surjective. Up to now we have just checked that our stack is Artin , we haven't used the condition on the finiteness of the stabilizers. Now we proceed in two directions: for the general case, we use the proposition 8.1 of [LM], and then we work out explicitly a subexample, i.e. weighted projective spaces.

There is a natural bijective correspondence between isomorphism classes of morphisms $Spec(\mathbb{K}) \longrightarrow [X/G]$ with orbits of geometric points $Y \longrightarrow X$. ih x_1 and x_2 are in $[X/G](Spec(\mathbb{K}))$, then $\mathrm{Isom}_{Spec(\mathbb{K})}(x_1, x_2)$ is empty unless x_1 and x_2 correspond to the same orbit, in which case it is isomorphic to the scheme-theoretic stabilizer of a point in the orbit. This implies that the diagonal is unramified. Then $[X/G]$ is an algebraic Deligne Mumford stack, according to the Proposition 8.1 [LM]. □

Example 5.4 (weighted projective space: check explicitly it is DM stack)**.** In this section $[X/G] = \mathbb{P}(w)$. Now it is enough to show that there exists a scheme U and an étale surjective morphism $u : U \longrightarrow [X/G]$. Let U_i be the usual finite open covering with affine charts of X . Let's consider the following object of $[X/G]$:

$$\begin{tikzcd} U_i \times G \arrow[r, "act_i"] \arrow[d, "\pi_i"'] & X \\ U_i & \end{tikzcd}$$

By 2-Yoneda's lemma, this object defines a morphism of stack $u_i : U_i \longrightarrow [X/G]$. We will show that this map u_i is in fact étale. Let (E, π, B, a) be an object of $[X/G]$. The lemma 5.2 implies that u_i factors through triv. The lemma 5.1 implies that the three square diagrams below are cartesian:

$$\begin{tikzcd} a^{-1}(U_i) \arrow[r, "\overline{incl}"] \arrow[d, "a\restriction_{a^{-1}(U_i)}"'] & P \arrow[r, "\pi"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ U_i \arrow[r, "incl"] & X \arrow[r, "triv"] & {[X/G]} \end{tikzcd}$$

where $u_i = triv \circ incl$. To show that the morphism of stacks u_i is étale, we will show that the morphism of schemes $\pi \circ \overline{incl} : a^{-1}(U_i) \longrightarrow B$ is étale. To be étale is a local condition on B in the Zariski topology, so we can assume that the principal bundle $\pi : P \longrightarrow B$ is trivial $P = X \times G$ and π is the projection on B. We have the following cartesian diagram:

$$\begin{tikzcd} a^{-1}(U_i) = X \times \mu_{w_i} \arrow[r] \arrow[d, "a|_X \times \mu_{w_i}"'] & X \times G \arrow[d] \\ U_i \arrow[r] & X \end{tikzcd}$$

where μ_{w_i} is the stabilizer. To finish the proof, we have just to show that the covering u given by the disjoint union of the u_i's is (étale) and surjective. This last condition, which can be checked on the associated reduced scheme X_{red} is trivial.

To end the section, we give the following lemma, which turns out to be fundamental in the last section:

Lemma 5.5. *Let G be a group scheme, acting on both $S \xrightarrow{f} T$ with finite stabilizers. Let f be $G-$equivariant. Then the induced morphism $[S/G] \xrightarrow{\tilde{f}} [T/G]$ is representable.*

Proof. We want to complete the diagram in order to make it cartesian:

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & {[S/G]} \arrow[d] \\ X \arrow[r] & {[T/G]} \end{tikzcd}$$

The morphism $X \longrightarrow [T/G]$ is given by an object:

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & T \\ X & \end{tikzcd}$$

$P \longrightarrow X$ is a $G-$principal bundle and π is $G-$equivariant. The morphism between the two quotient stacks is induced by the morphism $S \xrightarrow{f} T$ (the upper cartesian diagram is the definition of Q):

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ P \arrow[r, "\pi"] \arrow[d] & T \\ X & \end{tikzcd}$$

G acts over P without fixed points, hence it acts over Q without fixed points too:

$$g(p, s) \neq (p, s)$$

because $g(p) \neq p$. Hence we define the stack $Y = [Q/G]$. Let's see Y is a scheme. Taking a trivialization $\bigcup X_i = X$ for $P \longrightarrow X$, such that all X_i's are affine sets, one finds out the following diagram:

$$\begin{tikzcd} (G \times X_i) \times_T S \arrow[r] \arrow[d] & S \arrow[d] \\ G \times X_i \arrow[r] \arrow[d] & T \\ X_i & \end{tikzcd}$$

One now has to pass to the quotient of $(G \times X_i) \times_T S$ by the action of G. G only acts on the first and on the third variable, hence all can be done passing to the quotient in affine charts and then gluing. Now we have to prove that the diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & {[S/G]} \arrow[d] \\ X \arrow[r] & {[T/G]} \end{tikzcd}$$

is 2-cartesian. As usual, we check it only on objects, and we start proving it commutes. An object $U \longrightarrow Y$ is sent passing from the way up, to:

$$\begin{tikzcd} Q \times_Y U \arrow[r] \arrow[d] & T \\ U & \end{tikzcd}$$

Recall that $Q = P \times_X Y$. From the way above, one finds the following object:

$$\begin{tikzcd} U \times_X P \arrow[r] \arrow[d] & T \\ U & \end{tikzcd}$$

which is canonically isomorphic to $U \times_Y Y \times_X P$. Hence the diagram commutes.

Now, to see the diagram is 2-cartesian, take two morphisms from a scheme U to $[S/G]$ and to X respectively:

$$\begin{tikzcd} U \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & Y \arrow[r] \arrow[d] & {[S/G]} \arrow[d] \\ & X \arrow[r] & {[T/G]} \end{tikzcd}$$

Asking this diagram to be commutative means that the two principal bundles with $G-$equivariant morphism given by:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & T \\ U & \end{tikzcd} \qquad \begin{tikzcd} P \times_X U \arrow[r] \arrow[d] & T \\ U & \end{tikzcd}$$

are equivalent. We find out the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, dotted] \arrow[d, "\sim"] & Q \arrow[r] \arrow[d] & S \arrow[d] \\
P \times_X U \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & T \\
U \arrow[r] & X &
\end{tikzcd}$$

The upper right square is cartesian, hence we find out the dotted morphism. Now composing the dotted morphism with $Q \longrightarrow Y$ (the projection), we find the desired map, proving that our diagram was 2-commutative. In fact all the commuting rules follow, because the morphism factors through Q. Now $Q \longrightarrow Y$ is a $G-$principal bundle $\square$

Appendix B

Orbifolds and orbifold cohomology

1 Orbifolds

The notion of orbifold was first introduced by Satake in [Sa] under the name of *V-manifold*.

Definition 1.1 (1-ORB)**.** An *orbifold* is a smooth Deligne-Mumford Stack. A *morphism of orbifolds* is simply a morphism as algebraic stacks (and so a morphism as stacks).

For the differential definition of orbifold, we follow the description given in [Pe]. The standard reference is [MP], although it gives only the notion of reduced orbifold.

Let Y be a paracompact Hausdorff topological space. A *uniformizing system* for an open subset $U \subset Y$ is a collection of the following objects:

$\tilde{U}$ a connected open subset of $\mathbb{R}^d$;

G a finite group of C^∞-automorphisms of $\tilde{U}$ such that: the fixed-point set of each element of the group is either the whole space or of codimension at least 2, the multiplication in G is given by $g_1 \cdot g_2 = g_1 \circ g_2$ where $\circ$ is the composition;

χ a continuous map $\tilde{U} \to U$ that induces an homeomorphism from $\tilde{U}/G$ to U, where $\tilde{U}/G$ is the quotient space with the quotient topology. Here G acts on $\tilde{U}$ on the left.

We will call the subgroup of G which consists of elements fixing the whole space the *kernel* of the action, and it will be denoted by Ker(G).

Given an open subset U of Y, a uniformizing system for U will be denoted by $(\tilde{U}, G_U, \chi_U)$. If the dependence on U is clear from the context, it will also be denoted by $(\tilde{U}, G, \chi)$.

Definition 1.2. The *dimension* of an uniformizing system $(\tilde{U}, G, \chi)$ is the dimension of $\tilde{U}$ as a real manifold.

Let $(\tilde{U}, G, \chi)$ and $(\tilde{U}', G', \chi')$ be uniformizing systems for U and U' respectively, and let $U \subset U'$. An *embedding* between such uniformizing systems is a pair (φ, λ), where $\varphi : \tilde{U} \to \tilde{U}'$ is a smooth embedding such that $\chi' \circ \varphi = \chi$ and $\lambda : G \to G'$ is a group homomorphism such that $\varphi \circ g = \lambda(g) \circ \varphi$ for all $g \in G$. Furthermore, λ induces an isomorphism from Ker(G) to Ker(G′).

Definition 1.3 (2-ORB)**.** An *orbifold atlas* on Y is a family $\mathcal{U}$ of uniformizing systems for open sets in Y satisfying the following conditions:

1. The family $\chi(\tilde{U})$ is an open covering of Y, for $(\tilde{U}, G, \chi) \in \mathcal{U}$.
2. Let $(\tilde{U}, G, \chi), (\tilde{U}', G', \chi') \in \mathcal{U}$ be uniformizing systems for U and U' respectively, and let $U \subset U'$. Then there exists an embedding $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$.
3. Let $(\tilde{U}, G, \chi), (\tilde{U}', G', \chi') \in \mathcal{U}$ be uniformizing systems for U and U' respectively. Then, for any point $y \in U \cap U'$, there exists an open neighbourhood $U'' \subset U \cap U'$ of y and a uniformizing system $(\tilde{U}'', G'', \chi'')$ for U'' which belong to the family $\mathcal{U}$.

Two such atlases are said to be equivalent if they have a common refinement, where an atlas $\mathcal{U}$ is said to refine $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart in $\mathcal{V}$.

A *smooth orbifold* structure on a paracompact Hausdorff topological space Y is an equivalence class of orbifold atlases on Y.

We denote by $[Y]$ the smooth orbifold structure on the topological space Y. We will call this simply an orbifold.

Definition 1.4. The orbifold $[Y]$ is said of *real dimension* d if all the uniformizing systems of an atlas have dimension d. The real dimension of $[Y]$ will be denoted by $\dim_{\mathbb{R}}[Y]$ or, by abuse of notation by $\dim[Y]$.

Remark 1.5. Every orbifold atlas for Y is contained in a unique maximal one, and two orbifold atlases are equivalent if and only if they are contained in the same maximal one. Therefore we shall often tacitly work with a maximal atlas.

Proposition 1.6. *Let (φ, λ) and (ψ, μ) be two embeddings from $(\tilde{U}, G, \chi)$ to $(\tilde{U}', G', \chi')$. Then, there exists $g' \in G$ such that*

$$\psi = g' \circ \varphi \quad \text{and} \quad \mu = \mathrm{g}' \cdot \lambda \cdot \mathrm{g}'^{-1}.$$

Moreover, this g' is unique up to composition by an element of $\mathrm{Ker}(\mathrm{G}')$.

Remark 1.7. Notice that, for any uniformizing system $(\tilde{U}, G, \chi)$ and $g \in G$, there exists $g' \in G'$ such that $\varphi \circ g = g' \circ \varphi$. Moreover this g' is unique up to an element in $\mathrm{Ker}(\mathrm{G}')$. Thus, in the definition of an embedding (φ, λ), the existence of λ is required to guarantee a continuity for the kernels of the actions.

Remark 1.8. The previous remark implies that, for any embedding (φ, λ), the group homomorphism λ is injective.

Lemma 1.9. *Let $(\varphi, \lambda) : (\tilde{U}, G, \chi) \to (\tilde{U}', G', \chi')$ be an embedding. If $g' \in G'$ is such that $\varphi(\tilde{U}) \cap (g' \circ \varphi)(\tilde{U}) \neq \varnothing$, then g' belongs to the image of λ.*

Remark 1.10. Let $(\tilde{U}, G, \chi)$ be a uniformizing system for the open subset U of Y. Let $U' \subset U$ be an open subset. Then we have an *induced uniformizing system* $(\tilde{U}', G', \chi')$ for U', where $\tilde{U}'$ is a connected component of $\chi^{-1}(U')$ and G' is the maximal subgroup of G that acts on $\tilde{U}'$. Clearly there is an embedding of $(\tilde{U}', G', \chi')$ in $(\tilde{U}, G, \chi)$.

It follows that, for a given orbifold $[Y]$, we can choose an orbifold atlas $\mathcal{U}$ arbitrarily fine.

Remark 1.11. Let $[Y]$ be an orbifold and let $(\tilde{U}_1, G_1, \chi_1)$, $(\tilde{U}_2, G_2, \chi_2)$ be uniformizing systems for open subsets U_1, U_2 of Y in the same orbifold structure $[Y]$. For any point $y \in U_1 \cap U_2$, there is an open neighbourhood U_{12} of y such that $U_{12} \subset U_1 \cap U_2$, a uniformizing system $(\tilde{U}_{12}, G_{12}, \chi_{12})$ for U_{12}, compatible with $[Y]$, and embeddings $(\varphi_i, \lambda_i) : (\tilde{U}_{12}, G_{12}, \chi_{12}) \to (\tilde{U}_i, G_i, \chi_i)$ for $i \in \{1, 2\}$. So, we get the isomorphism:

$$\varphi_{12} := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(\tilde{U}_{12}) \to \varphi_2(\tilde{U}_{12}).$$

Let now U_1, U_2 and U_3 be open subsets of Y such that $U_1 \cap U_2 \cap U_3 \neq \varnothing$, and assume that there are uniformizing systems $(\tilde{U}_1, G_1, \chi_1)$, $(\tilde{U}_2, G_2, \chi_2)$ and $(\tilde{U}_3, G_3, \chi_3)$ for U_1, U_2 and U_3 respectively. Then, from Proposition 1.6, there exists $g \in G_3$ such that

$$\varphi_{23} \circ \varphi_{12} = g \circ \varphi_{13},$$

where the equation holds if we restrict the functions to some open subsets of the domains.

Definition 1.12. A *reduced* orbifold is an orbifold structure $[Y]$ on Y such that there exists an orbifold atlas $\mathcal{U}$ for $[Y]$ with the following property: for any uniformizing system $(\tilde{U}, G, \chi) \in \mathcal{U}$, Ker(G) is the trivial group.

Definition 1.13. Let $[Y]$ be a smooth orbifold and $y \in Y$ be a point. A *uniformizing system for* $[Y]$ *at* y is given by an open neighbourhood U_y of y in Y and a uniformizing system $(\tilde{U}, G, \chi)$ for U_y in the orbifold structure $[Y]$ such that, $\tilde{U} \subset \mathbb{R}^d$ is a ball centered in the origin $0 \in \mathbb{R}^n$, G acts trivially on 0 and $\chi^{-1}(y) = 0$.

For a given orbifold $[Y]$ and a point $y \in Y$, a uniformizing system at y will be denoted by $(\tilde{U}_y, G_y, \chi_y)$ and $\chi(\tilde{U}_y)$ by U_y. The group G_y will be also called the local group at y.

We now define a complex orbifold. We will use the same notation as in the smooth case.

Let Y be a paracompact Hausdorff topological space. A *complex uniformizing system* for an open subset U of Y is a triple $(\tilde{U}, G, \chi)$, where $\tilde{U} \subset \mathbb{C}^d$ is a connected open subset, G is a finite group of holomorphic automorphisms of $\tilde{U}$ and χ is a continuous map satisfying the same properties required in the smooth case.

Definition 1.14. The *complex dimension* of a complex uniformizing system $(\tilde{U}, G, \chi)$ is the dimension of $\tilde{U}$ as a complex manifold.

Let $(\tilde{U}, G, \chi)$ and $(\tilde{U}', G', \chi')$ be complex uniformizing systems for U and U' respectively, and let $U \subset U'$. A *complex embedding* between such uniformizing systems is a pair (φ, λ) satisfying the same properties stated in the smooth case but where $\varphi : \tilde{U} \to \tilde{U}'$ is holomorphic.

Definition 1.15. A *complex orbifold atlas* on Y is a family $\mathcal{U}$ of complex uniformizing systems for open sets in Y satisfying the conditions 1., 2. and 3. of Definition 1.3 where we replace embeddings with complex embeddings.

Two such atlases are said to be equivalent if they have a common refinement, where an atlas $\mathcal{U}$ is said to refine $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists a complex embedding into some chart in $\mathcal{V}$.

A *complex orbifold* structure on a paracompact Hausdorff topological space Y is an equivalence class of complex orbifold atlases on Y.

Definition 1.16. The complex orbifold $[Y]$ is said of *complex dimension* d if all the uniformizing systems of a complex atlas have complex dimension d. The complex dimension of $[Y]$ will be denoted by $\dim_{\mathbb{C}}[\mathrm{Y}]$ or, by abuse of notation by $\dim[\mathrm{Y}]$.

Remark 1.17. All the results given for the smooth case, holds in the complex case too. The notions of reduced complex orbifold and of uniformizing systems at a point are defined for complex orbifolds in the same way of the smooth case.

Lemma 1.18 (*Linearization lemma*,[Ca] Theorem 4.)**.** *Let $[Y]$ be a complex orbifold and let $y \in Y$ be a point. Then we can choose a local uniformizing system $(\tilde{U}_y, G_y, \chi_y)$ at y such that G_y acts linearly on $\tilde{U}_y$.*

We would like to give an idea of the equivalence of the two definitions, if one works over $\mathbb{C}$.

Theorem 1.19. *Let $[Y]$ be an orbifold. Any atlas $\mathcal{U}$ of $[Y]$ determines a groupoid which represents $[Y]$*

We now make a little digression on groupoids inside categories. The following is taken from [G] and [Vi1].

Remark 1.20. Let $\mathcal{C}$ be a (small) category. The axioms of the category give us four maps of sets:

$$\mathcal{MOR} \rightrightarrows \mathcal{OBJ} \xrightarrow{e} \mathcal{MOR}$$

$$\mathcal{MOR} \times_U \mathcal{MOR} \xrightarrow{m} \mathcal{MOR}$$

where the two parallel arrows give the source and target for each morphism and will be called s and t, e gives the identity morphism, and m is composition of morphisms. In the following, we will denote by U the set of objects and by R the set of morphisms.

If the category turns out to be a groupoid, then there is a fifth morphism:

$$R \xrightarrow{i} R$$

that gives the inverse. These maps satisfy the following (trivial) four properties:

1. $s \circ e = t \circ e = Id_R$, $s \circ i = t$, $t \circ i = s$, $s \circ m = s \circ p_2$, $t \circ m = t \circ p_1$ where p_1 and p_2 are respectively the first and the second projection.

2. $m \circ (m \times Id_R) = m \circ (Id_R \times m)$

3. both compositions:

$$R = R \times_{s,U} U = U \times_{U,t} R \xrightarrow[e \times Id_R]{Id_R \times e} R \times_{s,U,t} R \xrightarrow{m} R$$

   are equal to the identity map on R

4. $m \circ (i \times Id_R) = e \circ s$ and $m \circ (Id_R \times i) = e \circ t$.

This definition will be useful later:

Definition 1.21. Let $\mathcal{C}$ be a category with fiber products. A groupoid G inside $\mathcal{C}$ is the datum of:

- Two objects, U and R of $\mathcal{C}$.
- Five morphisms (s, t, i, m, e) satisfying the previous four prescriptions.

Definition 1.22. A *groupoid space* is a pair of spaces (sheaves of sets) U, R together with five morphisms s, t, i, m, e with the same properties as above once computed over an open set.

Lemma 1.23. *Let F be an algebraic stack with an atlas $U \longrightarrow F$, and set $R = U \times_S U$. Then $R \rightrightarrows U$ has a natural groupoid structure in which the structure morphisms are étale, and the diagonal $R \longrightarrow U \times_S U$ is quasicompact and separated.*

Proof. The unit morphism $U \longrightarrow R$ is the diagonal, the inverse $R \longrightarrow R$ switches the two factors and the composition:

$$R \times_U R = U \times_F U \times_F U \longrightarrow U \times_F U = R$$

is the projection onto the first and third factor. □

This groupoid is usually called a *presentation* of F. Choosing different atlases means choosing different presentations. Conversely, you can easily prove the following

Lemma 1.24. *Given a groupoid space, there is a pseudofunctor F whose objects are elements of the set $U(B)$ and whose morphisms over B are elements of the set $R(B)$. Given $f : B \longrightarrow B'$ the functor $F(f)$ is defined using the maps $(U(B) \longrightarrow U(B')$ and $R(B) \longrightarrow R(B')$.*

This procedure turns out to give a pseudofunctor which satisfies only the first stack condition but not the second (it is usually called a prestack). There is a general procedure to associate to a prestack a stack, which can be found in [LMB00] [2.4.3]. This is quite similar to the sheafification of a presheaf.

What follows is an adapted version of this, taken from [Vi1] (there are lot of misprintings pp 668-669).

Theorem 1.25. *Let there be given an étale groupoid $R \rightrightarrows U$ in $\mathcal{SCH}/\mathbb{C}$ in which the diagonal is quasicompact and separated. We can define a quotient stack F with an étale surjective map $U \longrightarrow F$ such that $R = U \times_F U$. Hence F turns out to be an algebraic stack with $R \rightrightarrows U$ as a presentation.*

Remark 1.26. If $F' \longrightarrow F$ is a substack of a stack F, $R \rightrightarrows U$ a presentation of F given by p_1 and p_2, and $U' := F' \times_F U$, then $p_1^{-1}(U') = p_2^{-1}(U')$. Conversely, given a subscheme U' of U, such that $p_1^{-1}(U') = p_2^{-1}(U')$, setting $R' = p_1^{-1}(U')$, $R' \rightrightarrows U'$ has a groupoid structure induced by the groupoid structure on $R \rightrightarrows U$, and it defines a substack F' of F which is open or closed if U' is open or closed in U.

Lemma 1.27 ([Kt])**.** *Let $f : X \longrightarrow Y$ be a map of algebraic spaces which is locally quasifinite, locally of finite presentation and separated. Suppose Y is a scheme, then X is a scheme.*

Conversely if $R \rightrightarrows U$ is a presentation of an algebraic stack, then by the procedure above we get a stack which is canonically isomorphic to F.

Theorem 1.28 ([EV] Corollary 2.16, Theorem 2.18, Corollary 2.19)**.** *Each orbifold (DM smooth stack over $\mathcal{SCH}_{\mathbb{C}}$) is a global quotient $[X/G]$.*

This justifies all the computations made at the end of chapter 3.

We conclude the section with a theorem which is somehow part of the folklore. For this there is no standard reference, the only work where we found it is [Pe].

Moreover, we didn't gave the notion of morphism and natural equivalence in the differential definition of orbifold. This can be found in [Sa], [CR1], or [Pe].

Theorem 1.29. *([Pe]) Over $\mathbb{C}$, the 2-category of (1-ORB) is equivalent to the 2-category of (2-ORB).*

2 Cohomology

This is taken from [B], which gives a much more complete description of cohomology, De Rham cohomology, homology and equivariant homology and cohomology for groupoids in some categories (topological spaces or differentiable manifolds, we will use it w.r.t. stacks on schemes with the etále topology). Here we study stacks via presenting groupoids. In this section we won't need more than just topological stacks. So all stacks in this section arise from pseudofunctors from $\mathcal{TOP}$ in groupoids. Although this construction can be performed in general, what we have in mind is the underlying topological site of an algebraic scheme, or the site étale of a topological scheme, or, in case of smooth schemes over the complex numbers, its complex topological site. Recall that the singular chain complex of a topological space X, denoted by $C_\bullet(X)$, is the abelian group of formal integer linear combinations of continuous maps $\Delta_q \longrightarrow X$, where Δ_q is the standard $q-$simplex. The boundary map $d_i : \Delta_{q-1} \longrightarrow \Delta_q$ induces maps $d_i : Hom(\Delta_q, X) \longrightarrow Hom(\Delta_{q-1}, X)$, and $d : C_q(X) \longrightarrow C_{q-1}(X)$ defined by $d(\sigma) = \sum_{j=0}^{q} d_j(\sigma)$. The functor C from topological spaces to complexes is covariant, furthermore, since $d^2 = 0$, it is possible to define the homology functor, and to check that it is a covariant functor too. Anyway, we want to construct cohomology, and so we have to work somehow dually and reverse arrows. As far as we have the d operator in homology, we also have a d operator in cohomology taken dualizing all, i.e. taking $Hom(C_\bullet, \mathbb{Z})$ and $Hom(\delta, Id_{\mathbb{Z}})$. Let now F be a stack in the category (for example) of topological spaces. Let $X_1 \rightrightarrows X_0$ be a presentation groupoid for the stack. It is now possible to build the double intersection space, defined as the fiber product of the following diagram:

$$\begin{tikzcd} & X_1 \arrow[d] \\ X_1 \arrow[r] & X_0 \end{tikzcd}$$

Again, it is possible to define $X_3 = X_1 \times_{X_0} X_1 \times_{X_0} X_1$,... and so on, defining X_p. There are $p+1$ differentiable maps $\partial_i : X_p \longrightarrow X_{p-1}$, given by leaving out the $i-th$ object. More precisely, ∂_i maps the element:

$$x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_p} x_p$$

to

$$x_0 \xrightarrow{\phi_0} x_1 \xrightarrow{\phi_1} \cdots \longrightarrow x_{i-1} \xrightarrow{\phi_i \circ \phi_{i+1}} x_i \longrightarrow \cdots \xrightarrow{\phi_p} x_p$$

This data will be summarized by the following diagram of topological spaces:

$$\begin{tikzcd} X_3 \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X_1 \arrow[r, shift left=1] \arrow[r, shift right=1] & X_0 \end{tikzcd}$$

It is now possible to form two complexes: $C_\bullet(X)$ and its dual complex $C^\bullet(X)$. Thus we have

$$C^n(X) = Hom(C_n(X), \mathbb{Z})$$

We get a diagram:

$$\begin{tikzcd} C^\bullet(X_3) & C^\bullet(X_2) \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & C^\bullet(X_1) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & C^\bullet(X_0) \arrow[l, shift left=1] \arrow[l, shift right=1] \end{tikzcd}$$

By defining $\partial = \sum_{i=0}^{p} (-1)^i \partial_i$ we get a morphism of complexes $\partial : C_\bullet(X_p) \longrightarrow C_\bullet(X_{p-1})$. Thus there is defined a double complex:

$$\begin{tikzcd} \cdots & C_0(X_2) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_0(X_1) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_0(X_0) \arrow[l, "\partial"'] \arrow[d, "d"'] \\ \cdots & C_1(X_2) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_1(X_1) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_1(X_0) \arrow[l, "\partial"'] \arrow[d, "d"'] \\ \cdots & C_2(X_2) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_2(X_1) \arrow[l, "\partial"'] \arrow[d, "d"'] & C_2(X_0) \arrow[l, "\partial"'] \arrow[d, "d"'] \\ & \cdots & \cdots & \cdots \end{tikzcd}$$

The associated total complex $C_\bullet(X)$ is then defined as:

$$C_n(X) = \bigoplus_{p+q=n} C_q(X_p)$$

with the differential $\delta : C_n(X) \longrightarrow C_{n-1}(X)$ given by:

$$\delta(\sigma) = (-1)^{p+q}\partial(\sigma) + (-1)^q d(\sigma)$$

It is possible to check that $\delta^2 = 0$.

Definition 2.1. The complex $(C^\bullet(X), \delta)$ is called the *singular cochain complex* of the topological groupoid $X = [X_1 \rightrightarrows X_0]$. Its cohomology groups, denoted $H^n(X;\mathbb{Z})$ are called the *singular cohomology groups* of $X_1 \rightrightarrows X_0$.

Theorem 2.2. *[B] If the space X is contractible, and the groupoid is given by $G \rightrightarrows X$, then the cohomology is just the group cohomology of the group G, taken with $\mathbb{Z}$ coefficients.*

Theorem 2.3. *[B] If you change the representing groupoid, cohomology doesn't change.*

This is proved thanks to invariance of cohomology under Morita Equivalence, at least in the smooth case.

We work always on Deligne-Mumford stacks considering them just from their topological viewpoint.

Proposition 2.4. *([B], Exercise 34) Let $X_1 \rightrightarrows X_0$ be two topological groupoids representing a topological DM stack F. Then X_0 can be covered by open subsets U_i, such that the restriction of the groupoid $X_1 \rightrightarrows X_0$ to U_i is a transformation groupoid $G_i \times U_i \rightrightarrows U_i$ for a finite group G_i acting on U_i for all i. We say that a topological DM stack is locally a finite group quotient.*

Recall that the image of the diagonal $X_1 \longrightarrow X_0 \times X_0$ is an equivalence relation on X_0. In fact, the assumption onto properness of the diagonal, guarantees that this equivalence relation is closed, and hence admits a Hausdorff quotient space $\overline{X}$.

Proposition 2.5. *The space $\overline{X}$ depends only on the Morita equivalence and hence is an invariant of the associated topological stack F. This is called the coarse space of F.*

Theorem 2.6. *(Deligne-Mumford stacks have coarse moduli spaces) [[KM] Deligne-Mumford algebraic stacks have coarse moduli spaces, whose support set is just the geometric quotient of the space by the groupoid.*

Keel and Mori proved that the topological quotient which is coarse moduli space for the considered stack, is a scheme which is an algebro-geometric quotient.

For example, the coarse space of the stack BG, for a finite group G, is the point $\{*\}$. The singular cohomology of a stack and its coarse space can be really different: the cohomology of BG is group cohomology, whereas the cohomology of the point is trivial.

The difference between the cohomology of the DM stack and its coarse space is entirely due to torsion phenomena:

Theorem 2.7. *Let F be a topological DM stack with coarse space $\bar{F}$. Then the canonical morphism $F \longrightarrow \bar{F}$ induces isomorphisms on $\mathbb{Q}$-valued cohomology groups:*

$$H^k(\bar{F}, \mathbb{Q}) \xrightarrow{\cong} H^k(F, \mathbb{Q}) .$$

3 The Inertia Stack

This is a natural stack associated to a stack F, which in a way points to where F fails to be a space.

Definition 3.1. Let F be a stack. The Inertia Stack I_F of F is defined as follows. If T is a scheme over S, an object of $I_F(T)$ consists of an object τ of $F(T)$ and an automorphism of τ in $F(T)$. The arrows in I_F are arrows in F compatible with the automorphisms.

Remark 3.2. [Vi1] An alternate description of I_F is as the fiber product $F \times_{F \times_S F} F$ where both morphisms $F \longrightarrow F \times_S F$ are the diagonal. To simplify the notations, we assume that $S = Spec_{\mathbb{K}}$ (this is the case we are really interested in). Our realization of the fiber product in 3.13 prescribes that we have as objects triples (X, Y, α), where X and Y are objects in $F(A)$ and $\alpha = (\alpha_1, \alpha_2) : (X, X) \longrightarrow (Y, Y)$ and $\alpha : (X, X) \longrightarrow (Y, Y)$ is an automorphism which lies in $F \times F(Id_A)$. We now have two possible choices: to take $(X, \alpha_1^{-1} \circ \alpha_2)$ or to take $(Y, \alpha_2^{-1} \circ \alpha_1)$. This two choices give rise to two isomorphic results, in fact switching from one description to the other is the inertia stack involution. Conversely, given a couple (X, α) one gets an element of the fiber product taking $(X, X, \bar{\alpha})$, where $\bar{\alpha} = (\alpha, Id)$ or, equivalently, $\bar{\alpha} = (Id, \alpha)$. We have shown how the functor acts on objects, and leave the reader to check it on morphisms.

To conclude the remark, observe that from this second description it turns out easily that the inertia stack of a Deligne-Mumford algebraic stack is itself in a natural way a Deligne-Mumford algebraic stack.

Remark 3.3. I_F is a group stack over F as follows. Consider the functor $I_F \longrightarrow F$ obtained by forgetting the automorphisms. There is a product $I_F \times_F I_F \longrightarrow I_F$ resulting from composing the automorphisms.

Lemma 3.4 ([Vi1]). *1. The category I_F is an algebraic stack, and the morphism $I_F \longrightarrow F$ is representable, separated, quasifinite and unramified.*

2. If there exists a scheme M and a separated morphism of finite type from F to M, then $I_F \longrightarrow F$ is finite.

Remark 3.5. When considering schemes of finite type over $\mathbb{K}$, the second condition is always satisfied (with $Spec(\mathbb{K})$ as M).

Proposition 3.6. *Let F be a Deligne-Mumford stack. Then the inertia stack I_F has a connected component which is isomorphic to F. This is usually called untwisted sector, and all other connected components are usually called twisted sectors.*

Let's now consider the case of a global quotient $[X/G]$. We want to prove the following formula for the inertia stack of $[X/G]$ (this is well known, but we weren't able to find such a result anywhere):

Proposition 3.7.

$$I([X/G]) = \coprod_{(g)\in T} [X^{(g)}/C(g)] = [G \times X/G]$$

where we denote by T a set of one representative element for each conjugacy class in the group (arbitrarily chosen), (g) is one element in it and $C(g)$ is the centralizer of the element g inside the finite group G.

Here in $[G \times X/G]$, G acts on G by conjugation and on X with the usual action.

Proof. We start from the particular case of the quotient of a point by the (trivial) action of a finite group on it. Second simplification: we show that the two pseudo-functors of the proposition give equivalent categories when applied to a closed point.

Now, the objects of the first category are couples (P, α): an homogeneous $G-$spaces ($G-$torsor) together with an automorphism.

Take an element $y \in P$, $\alpha(y) = g_1 y$ for some $g_1 \in G$ (transitivity of the action). If $x = gy$, then observe that:

$$g_1 g^{-1} x = g_1 y = \alpha(g^{-1}x) = g^{-1}\alpha(x)$$

Now take $P^{(g)} := \{gy| \ \ g \in C(g_1)\} = C_{g_1}y$. This is clearly a $C(g_1)-$torsor, since the action of α on $P^{(g)}$ is:

$$\alpha(gy) = g\alpha(y) = gg_1\alpha(y) = g_1 gy$$

(α acts as automorphism exactly as the left multiplication by the element g_1). This gives us a $C(g_1)-$ torsor. This functor F_y (we stress it depends on y) is fully faithful, in fact the automorphisms of (P, α) are morphisms of the $G-$torsor P commuting with the action of α, hence $\beta \longrightarrow F_y(\beta)$ is a generic element of the group G commuting with g_1, hence an element of $C_(g_1)$.

Let's show an inverse of such a functor. An element of

$$\coprod_{(g)\in T} BX(g)(\mathrm{Spec}(\mathbb{K}))$$

is a $C(g)-$torsor Q. Take $Q \times G$ and then take the quotient via the action of $C(g)$:

$$(q, h) \longrightarrow (aq, a^{-1}h)$$

We have a natural G action on this quotient: $(q, h) \longrightarrow (q, hG)$. This action clearly passes to the quotient. As automorphism, we take the multiplication by the chosen element g. The centralizer of a generic point $y \in C(g) \times Q/\sim$, $y \longrightarrow \alpha(y) = gy$ is $C(g)$.

This gives the desired correspondence F_y when one computes pseudofunctors on points. In a similar way one gets the same result when computing it on a generic scheme S. In fact, if $s \in S$ is a geometric point, one takes y in the fiber of the given $G-$principal bundle P (in fact, a Galois covering) and then repeats the same argument choosing a local section of the bundle $s \longrightarrow y(s)$ and then noticing that the association $y(s) \longrightarrow F_y(\alpha)$ gives a morphism of a neighbourood of s into the group G which is discrete. Hence the association is locally constant, and so it is constant on connected components of S. Finally, one has to repeat the argument for any connected component of S.

Now, for the case of the stack $[X/G]$ one has to do the same taking also in account the equivariant morphism to X. Taking in account our simplification, we compute all on a geometric point. So one has to send a trivial G torsor with $G-$equivariant morphism π to a $C(g)-$torsor with a $C(g)-$equivariant morphism π_g which takes values in $X^{(g)}$, and conversely. Now the automorphism α is an automorphism of the $G-$torsor together with the $G-$equivariant map π, hence it satisfies $\pi\alpha = \pi$. So on $y' \in C(g_1)y$ $\pi(y') = \pi(gy') = g\pi(y')$, hence the image of $C(g_1)$ via π is $G-$equivariant. So we simply send the map π to its restriction to $C(g_1)y$, to get a $C(g_1)$-equivariant map to $X^{(g)}$. Conversely, given the $C(g)$ torsor with π_g a $C(g)-$ equivariant morphism, one gets a $G-$equivariant morphism from $(Q \times G)/\sim \longrightarrow X$ which extends this one, taking as π:

$$[(q, h)] \longrightarrow \pi_g(q)h$$

□

4 Orbifold Cohomology as graded vector space

Definition 4.1. (Orbifold Cohomology as vector space)

For this section we refer to [CR1] for the differential definition. For the definition in the algebraic context, see [A], [AGV] and [AV].

Let A be a commutative ring. We define:

$$H^*_{orb}(M, A) := H^*(I(M), A)$$

as $A-$module. Althought it is in general interesting to compute it for $A = \mathbb{Z}$, in the following we will choose the coefficients in $\mathbb{Q}$, to have that the cohomology of a twisted sector is just its coarse space's singular cohomology.

We review now the definition of degree shifting. Let $y \in Y$ be any point and let $(\tilde{U}_y, G_y, \chi_y)$ be a uniformizing system at y. The origin 0 of $\tilde{U}_y$ is fixed by the action of G_y, so we have an action of G_y on the tangent space of $\tilde{U}_y$ at 0. We represent this action by a group homomorphism $R_y : G_y \to GL(d, \mathbb{C})$, where $d = \dim_{\mathbb{C}} \mathrm{Y}$. For every $g \in G_y$, $R_y(g)$ can be written as a diagonal matrix:

$$R_y(g) = \mathrm{diag}\left(\exp(2\pi \mathrm{i} \mathrm{m}_{1,\mathrm{g}}/\mathrm{m}_\mathrm{g}), ..., \exp(2\pi \mathrm{i} \mathrm{m}_{\mathrm{d},\mathrm{g}}/\mathrm{m}_\mathrm{g})\right)$$

where m_g is the order of $R_y(g)$, and $0 \leq m_{i,g} < m_g$ is an integer. Since this matrix depends only on the conjugacy class $(g)_y$ of $g \in G_y$, we define a function $a : Y_1 \to \mathbb{Q}$ by

$$a(y,(g)_y) = \sum_{i=1}^{d} \frac{m_{i,g}}{m_g}$$

For the proof of the following lemma the notion of tangent sheaf (or tangent bundle) to an algebraic stack (or respectively, to an orbifold) is needed. We do not enter these details, since we just need the assertion:

Lemma 4.2. *([CR1] Lemma 3.2.1) For any $(g) \in T$ the function $a : Y_{(g)} \to \mathbb{Q}$ is constant on each connected component. Let n be the (complex) dimension of X. Then one has, for $x \in X_{(g)}$:*

$$a(g,x) + a(g^{-1},x) = n - dim(X_{(g)})$$

Definition 4.3. For any $(g) \in T$, the *degree shifting number* of (g) is the locally constant function

$$a(y,(g)_y) : Y_{(g)} \to \mathbb{Q}.$$

If $Y_{(g)}$ is connected, we identify $a(y,(g)_y)$ with its value, and we will denote it by $a_{(g)}$ too.

Remark 4.4. In the literature, the degree shifting number is also known as *age*.

Definition 4.5. For any integer p, the degree p *orbifold cohomology group* of $[Y]$, $H^p_{orb}([Y])$, is defined as follows

$$H^p_{orb}([Y]) = \oplus_{(g)\in T} H^{p-2a_{(g)}}(Y_{(g)})$$

where $H^*(Y_{(g)})$ is the singular cohomology of $Y_{(g)}$ with complex coefficients. The *total orbifold cohomology group* of $[Y]$ is

$$H^*_{orb}([Y]) = \oplus_p H^p_{orb}([Y]).$$

Remark 4.6. Note that $H^*_{orb}([Y])$ is a priori rationally graded. It is integrally graded if and only if all the degree shifting numbers are half-integers.

We now give some example of simple orbifold cohomology with integer coefficients.

Example 4.7. Since the integer coefficients case depends on group cohomology of all centralizers, we give the example of:

$$H^*_{\text{orb}}(BG, \mathbb{Z})$$

With G an abelian group. The inertia stack is:

$$\coprod_{g \in G} BG_g$$

(a number of copy of BG equal to $\#G$). The structure theorem for abelian groups allows us to reduce to the case where $G = \mu_r$. Since the group cohomology of μ_r is $\mathbb{Z}[t]/(rt)$, and there is no age (we are in dimension 0), we can write the cohomology (as $\mathbb{Z}$-module):

$$H^*_{\text{orb}}(B\mu_r, \mathbb{Z}) = \mathbb{Z}[s,t]/(s^r - 1, rt)$$

where the grading is given only by t.

Example 4.8. The twisted affine line.

Consider the global quotient: $[\mathbb{A}^1/\mu_r]$, where the group act in the standard way. We can explicitly write:

$$I([\mathbb{A}^1/\mu_r]) = [\mathbb{A}^1/\mu_r] \coprod_{k=1}^{r-1} B\mu_r$$

An easy calculation shows that the age of X_i is i/r.

Appendix C

Smooth and stable curves

1 Definition and first properties

All schemes will be of finite type over $\mathbb{K}$, with $\mathrm{char}(\mathbb{K}) = 0$.

Definition 1.1. A *curve*, is a reduced scheme C of dimension 1. If not otherwise stated, every curve is supposed to be connected. The curve is *smooth* (*regular*) whenever $\mathcal{O}_{x,C}$ is a regular local ring.

Recall that for curves (and if Cohen Macaulay more generally in codimension 1) regular and normal is the same:

Lemma 1.2 (cf. [AM] proposition 9.2)**.** *Let A be a Noetherian local domain of dimension one, m its maximal ideal, $\mathbb{K} = A/m$ its residue field. Then the following are equivalent:*

1. *A is integrally closed;*
2. *m is a principal ideal;*
3. *$dim_{\mathbb{K}}(m/m^2) = 1$;*
4. *every non-zero ideal is a power of m;*
5. *there exists $x \in A$ such that every non-zero ideal is of the form (x^k), $k \geq 0$.*

Definition 1.3. (Genus) Let C be a projective curve. Its *arithmetic genus* is $\dim_{\mathbb{K}}(H^1(C,\mathcal{O}_C))$. If the curve is smooth, its *geometric genus* is $\dim_{\mathbb{K}}(H^0(C,\omega_C))$. Moreover, if $\mathbb{K} = \mathbb{C}$, the *topological genus* is $\frac{1}{2}\dim_{\mathbb{C}}(H^1(C,\mathbb{C}))$, where this last cohomology group is taken with the strong (analytic) topology on C.

Theorem 1.4 (cfr. [H] chap. 3.7)**.** *(Serre Duality) Let X be a projective Cohen-Macaulay scheme of equidimension n over $\mathbb{K}$. Then for any locally free sheaf $\mathcal{F}$ on X, there are natural isomorphisms:*

$$H^i(X,\mathcal{F}) \cong H^{n-i}(X,\mathcal{F}^* \otimes \omega_X^c)^*$$

where ω_X^c is the dualizing sheaf. If X is smooth, then $\omega_X^c = \Omega_X^n$ where the last is the canonical sheaf.

Theorem 1.5. *(Hodge Decomposition) [See [GH] chap. 0.6,0.7] Let X be a smooth projective complex variety. Then:*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^p(X,\Omega_C^q)$$

Corollary 1.6. *Let C be a projective curve. In case it is possible to define two genera, they turn out to be the same number.*

Proof. By Serre duality, $p_a = p_g$. By Hodge decomposition, $H^1(C,\mathbb{C}) = H^1(C,\mathcal{O}_C) \oplus H^0(C,\Omega_C^1)$ □

Definition 1.7 (cfr. [H] ex. 1.5.3)**.** Let f be a polynomial, and let $Y \subset \mathbb{A}^2$ be a curve defined by the equation $f(x,y) = 0$. Write f as a sum $f = f_0 + f_1 + ... + f_d$ where f_i is a homogeneous polynomial of degree i in x and y. The *multiplicity* of 0 on Y is the least r such that $f_r \neq 0$. The linear factors of f_r are called *tangent directions* at 0.

Definition 1.8. A *node* (also called *ordinary double point*) is a double point (a point of multiplicity 2) with distinct tangent directions.

Definition 1.9. A *projective nodal curve* is a projective curve whose points are either smooth or nodes. A *projective nodal curve with n marked points* has in addition a choice of n ordered, distinct regular points.

Definition 1.10. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X. The *Euler characteristic* of $\mathcal{F}$ is:

$$\chi(\mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{K}} H^i(X, \mathcal{F})$$

Remark 1.11. χ is an additive function ([H] chap. 3.5 ex 5.1). By the vanishing theorem of Serre [[H], Theorem 5.17], one has $H^i(X, \mathcal{O}_X(m)) = 0$ for all $i > 0$ for $m >> 0$.

Theorem 1.12 ([H] chap. 4.1 ex.1.9)**.** *(Riemann-Roch for projective nodal curves) Let C be a (possibly singular) projective curve, and let D be a divisor supported in the set of regular points of C. Denoting by $\mathcal{L}(D)$ its associated invertible sheaf, the following formula holds:*

$$\chi(\mathcal{L}(\mathcal{D})) = deg(D) + 1 - p_a$$

Remark 1.13. If the curve is nodal, then the dualizing sheaf is an invertible sheaf (cfr. [H] chap 3, 7.11). Hence one can define the canonical divisor K to be a divisor with support in the set of regular points of C, corresponding to ω_C. Calling $l(D)$:=dim$(H^0(C, \mathcal{L}(D)))$, the previous formula (also by Serre duality) becomes:

$$l(D) - l(K - D) = deg(D) + 1 - p_a$$

If the curve is smooth, one finds out the usual formulas $l(K) = g$, hence $deg(K) = 2g - 2$, furthermore if $deg(D) > 2g - 2$, then $deg(K - D) < 0$ and so $l(K - D) = 0$ (one says that D is nonspecial).

To the divisor D, one can associate the complete linear system $|D|$, all the effective divisors linearly equivalent to D. $|D|$ is naturally identified with $\mathbb{P}(\mathcal{L}(D))$.

If a divisor D on C is given, a necessary and sufficient condition for $|D|$ for being basepoint free is that $l(D-P) = l(D)-1$ for all points $P \in C$ (cfr. [H] chap. 4 prop. 3.1). if $|D|$ is basepoint free, a base of $\mathcal{L}(D)$ defines a morphism $\phi_D : C \longrightarrow \mathbb{P}^n$, where $n = l(D) - 1$. A different base of $\mathcal{L}(D)$ gives a morphism which differs from the previous one by a projectivity.

Proposition 1.14 (cfr. [H] chap. 2, 7.3)**.** *Let D be a divisor on the smooth curve C, with $|D|$ base point free and ϕ_D a morphism associated to $\mathcal{L}(D)$. Then ϕ_D is an isomorphism onto its image iff it separates points and tangent vectors, i.e. for all*

$P \in C$, ϕ_D and $d_p\phi_D : T_P(C) \longrightarrow T_{\phi_D(P)}(\phi_D(C))$ are injective. This condition is equivalent to the following: for all $P, Q \in C$, possibly $P = Q$:

$$l(D - P - Q) = l(D) - 2$$

(a divisor with this last property is usually called very ample)

If the divisor D had degree greater than $2g$, its complete linear system is basepoint free. In particular, if $g = 1$, each divisor of degree ≥ 3 is very ample.

Lemma 1.15. *Let C be the disjoint union of n connected components C_i. One has:*

$$g_C = \sum_{i=1}^{n} g_{C_i} - n + 1$$

Proof. $\chi(\mathcal{O}_C) = 1 - g$. By RR for $\mathcal{O}_C(m)$:

$$\chi(\mathcal{O}_C(m)) = m \deg(\mathcal{O}_C(1)) + 1 - g$$

Is C is the disjoint union: $C_1 \coprod C_2$:

$$H^0(C, \mathcal{O}_C(m)) = H^0(C_1, \mathcal{O}_{C_1}(m)) \oplus H^0(C_2, \mathcal{O}_{C_2}(m))$$

if m is great enough, these are the Euler characteristic:

$$\chi(\mathcal{O}_C(m)) = \chi(\mathcal{O}_{C_1}(m)) + \chi(\mathcal{O}_{C_2}(m))$$

It turns out that $g_C = g_{C_1} + g_{C_2} - 1$, the formula now follows by induction. □

Proposition 1.16. *Let $\nu : (\tilde{C}) \longrightarrow C$ be the normalization of an irreducible nodal curve. Then:*

$$g_C = g_{\tilde{C}} + n$$

where n is the number of nodes of C.

Proof. Recall ν is a finite, surjective, birational morphism. Hence $\nu_*(\mathcal{O}_{\tilde{C}})$ is a coherent sheaf. We have the following exact sequence of sheaves, defining $\mathcal{Q}$

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{\tilde{C}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The sheaf $\mathcal{Q}$ is supported on the singular locus of C, since where C is smooth the morphism ν is an isomorphism. If p is a node and U is a neighbourood of p without other singularities than p, then $\mathcal{Q}_{|U} \cong \mathcal{O}_p$. Hence, being C a nodal curve:

$$\mathcal{Q} \cong \bigoplus_{p \text{ node}} \mathcal{O}_p$$

furthermore, since ν is finite, $\chi(\mathcal{O}_{\tilde{C}}) = \chi(\nu_*\mathcal{O}_{\tilde{C}})$. So, following [H] ex. 2.3.8, 2.5.17, 3.4.1, it follows $H^i(\tilde{C}, \mathcal{O}_{\tilde{C}}) \cong H^i(C, \nu_*\mathcal{O}_{\tilde{C}})$, so one finds:

$$\chi(\mathcal{O}_{\tilde{Y}}) = \chi(\mathcal{O}_Y) + \sum_{p \text{ node}} \chi(\mathcal{O}_p) = \chi(\mathcal{O}_C) + n$$

where n is the number of nodes of C. Now the usual relation between Euler characteristic and genus (for curves) gives the desired result. □

Remark 1.17. It is a conventional notation to define the geometric genus as the geometric genus of the normalization (it is smooth). For a nodal curve, geometric genus and arithmetic genus differ by the number of nodes. Remember that $g = 0 \implies$ birational to $\mathbb{P}^1$.

Corollary 1.18. *If C is a connected non irreducible curve with n nodes, and r irreducible components, and if $\tilde{C}$ is its normalization, it has r distinct connected components. Then:*

$$g_C = \sum g_{\tilde{C}} + n - r + 1$$

Since the curve is connected, $n \geq r - 1$, hence

$$g_C \geq \sum g_{\tilde{C}_i}$$

in particular if C is a genus zero nodal curve, then $n = r - 1$.

Proposition 1.19. *Let C be a nodal and arithmetic genus zero curve. Then every irreducible component of C is smooth.*

Proof. Normalizing only intersection points, one finds a scheme Y and a canonical projection onto C:

$$g_C = \sum g_{C_i} + n - r + 1$$

normalizing each C_i, we find:

$$g_{C_i} = g_{\tilde{C}_i} + m_i$$

where m_i is the number of nodes on C_i. But $g_C = 0$, hence $g_{C_i} = 0$, and so all m_i's are 0. □

Definition 1.20. An *automorphism* of a scheme is an invertible morphism whose inverse is again a morphism. If $(C, x_1, ..., x_n)$ is a projective nodal curve with n marked points, an automorphism of it is just an automorphism which fixes all the marked points.

Definition 1.21. A projective nodal curve C with n marked points is said to be *stable* if the group $\mathrm{Aut}(C, x_i)$ of automorphisms of $(C, x_1, ..., x_n)$ is finite.

Theorem 1.22. *The following are equivalent:*

- $(C, x_1, ..., x_n)$ *is stable;*
- *Denoting by* T_C *the tangent sheaf of* C*,* $H^0(C, T_C(-\sum x_i)) = 0$*;*
- *every irreducible component of* C *of arithmetic genus* g *has at least* $3 - 2g$ *special points, i.e. intersection points or marked points.*

Proof. (sketch) $\mathrm{Aut}(C, x_i)$ is a group scheme, hence smooth. $\mathrm{Aut}(C, x_i)$ is finite if and only if it is a zero dimensional scheme. Being smooth, every tangent space has dimension zero. The tangent space to $\mathrm{Aut}(C, x_i)$ at the identity element is $H^0(C, T_C(-\sum x_i))$. □

Remark 1.23. If C is a smooth, connected, genus g curve, then:

1. if $g \geq 2$ then $\mathrm{Aut}(C) < \infty$. Actually also $|\mathrm{Aut}(C)| \leq 84(g-1)$ holds as a consequence of the Riemann-Hurwitz formula ([H], Chap 4, Ex. 2.5);
2. if $g = 1$ then the minimal number i of fixed points to have that $\mathrm{Aut}(C, p_1, ..., p_i) < \infty$ is one;
3. if $g = 0$ then the minimal number i of fixed points to have that $\mathrm{Aut}(C, p_1, ..., p_i) < \infty$ is three.

Let now C be an irreducible, genus g nodal curve. By $\tilde{C}$ we will denote its normalization.

Corollary 1.24. *A nodal curve* C *(without marked points) is stable if and only if the following hold:*

- *every irreducible component of arithmetic genus* 1 *of* C *meets at least one other component of* C*;*

- *every smooth irreducible rational component meets the other components in at least three points.*

Remark 1.25. A stable curve of genus zero with n marked points, has at most $n-2$ irreducible components.

2 Curves of genus 1

Definition 2.1. An elliptic curve is a pair $(E, 0)$, with E smooth connected projective curve of genus 1, and $0 \in E$ a point.

Remark 2.2. Every smooth planar cubic projective curve is elliptic. If it has a node, it is rational.

Theorem 2.3. *(Weierstrass embedding theorem) Every elliptic curve (smooth irreducible genus 1 curve with one marked point) (C, P) can be embedded in $\mathbb{P}^2_{\mathbb{K}}$ as a smooth cubic. There is a coordinate change which puts it in the affine form:*

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

where the α_i's are distinct elements of $\mathbb{K}$. The marked point is the (unique) point at infinity.

Proof. Let's study meromorphic functions with a pole in the marked point P, i.e. the linear spaces $L(nP)$. By RR $\dim(L(nP)) = n = \deg(nP)$ $L(P)$ is generated only by constant functions.
$L(2P)$ is spanned by (say) $\langle 1, x\rangle$. $\mathrm{ord}_p(x) = -2$.
$L(3P)$ is spanned by (say) $\langle 1, x, y\rangle$. $\mathrm{ord}_p(y) = -3$.
$L(3P)$ gives a closed embedding in $\mathbb{P}^2$, because it is very ample. To find out the relation between the generators, one looks at $L(4P)$ has dimension 4 and contains $\langle 1, x, x^2, y\rangle$
$L(5P)$ has dimension 5 and contains $\langle 1, x, x^2, y, xy\rangle$
$L(6P)$ has dimension 6 and contains $\langle 1, x, x^2, x^3, y, xy, y^2\rangle$, hence they are linearly dependent:

$$A_6 y^2 + A_5 xy + A_3 y = A_7 x^3 + A_4 x^2 + A_2 x + A_1 \tag{2.4}$$

observe that $A_6 A_7$ is different from zero, otherwise one has the sum of rational functions of different orders being zero (clearly impossible).

Now let's rescale the variables, substitute respectively x, y with $-A_6A_7x$ and $A_6A_7^2y$, and divide by $A_6^3A_7^4$ obtaining:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.5}$$

These x and y are called *Weierstrass coordinates.* Now an affine transformation:

$$x = x'y = y' - \frac{a_1}{2}x' - \frac{a_3}{2}$$

gives an expression without the xy, y terms. This is a cubic in the form given in the statement (since the field is algebraically closed). They are distinct since otherwise the curve would be singular. The point P is sent to $[0:1:0]$ since the order of y in P is strictly greater than the order of x. □

A simple translation on the x axis, permits us to write the elliptic curve in the form (which will always be used in the last chapter):

$$y^2 = x^3 + ax + b$$

Remark 2.6. $\alpha_1, \alpha_2, \alpha_3$ are distinct, in fact the curve is singular if and only if the following three equations are satisfied:

$$\begin{cases} y^2 - (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) & = & 0 \\ \frac{d}{dx}(y^2 - (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) & = & 0 \\ \frac{d}{dy}(y^2 - (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) & = & 2y & = & 0 \end{cases}$$

i.e. the three roots are different. Moreover, there is only one point at infinity, which turns out to be a flex point (the intersection with the infinity line $z = 0$ has multiplicity 3).

From now on, whenever no confusion arises, we will denote the elliptic curve with its Weierstrass representation.

Lemma 2.7. *(automorphisms) Let $C, 0$ be an elliptic curve with an associated Weierstrass equation. Then each automorphism ϕ of $(C, 0)$ is a coordinate change in $\mathbb{P}^2$ of the kind:*

$$\begin{cases} x = u^2x' + r \\ y = u^3y' + u^2sx' + t \end{cases}$$

with $u \neq 0$.

Proof. The automorphism ϕ induces isomorphism on $\mathcal{L}(2P)$, $\mathcal{L}(3P)$ which map respectively $\{1, x\}$ in $\{1, x'\}$ and $\{1, x, y\}$ in $\{1, x', y'\}$. Being $\{x, y\}$ and $\{x', y'\}$ Weierstrass coordinates, the following hold:

$$x = u_1 x' + r$$

$$y = u_2 y' + s_2 x' + t$$

for suitably chosen u_1, u_2, r, s_2, t with $u_1 u_2 \neq 0$. Since in the Weierstrass equation the coefficients of Y^2 and of X^3 are both 1, $u_1^3 = u_2^2$. Now letting $u := u_2/u_1$ and $s = s_2/u_1$ we have the equations of the statement. $\square$

Remark 2.8. (the Cross-Ratio) Let's recall some generalities about the cross-ratio. Given four points P_1, P_2, P_3, P_4 on $\mathbb{P}^1$ the cross ratio is the affine coordinate for P_4 in the projective frame given by P_1, P_2, P_3. Its (famous) formula is given by:

$$\beta([a_1 : b_1], [a_2 : b_2], [a_3 : b_3], [a_4 : b_4]) := \frac{(a_1 b_4 - a_4 b_1)(a_2 b_3 - b_3 b_2)}{(a_2 b_4 - a_4 b_2)(a_1 b_3 - a_3 b_1)}$$

It depends on the order of the four points considered. The permutation of 4 points are 24, but there are only six different cross-ratios:

$$\beta,\ \frac{1}{\beta},\ 1-\beta,\ \frac{1}{\beta-1},\ \frac{\beta}{\beta-1},\ \frac{\beta-1}{\beta}$$

since each of them is related to four distinct permutations (the stabilizer of the action of S_4 is the Klein V_4 group). We stress till now that for two particular values of β there are less different cross-ratios, i.e.:

$$\beta = -1, 1/2, 2 \qquad \beta = -e^{\frac{2\pi i}{3}}, -e^{\frac{4\pi i}{3}}$$

Definition 2.9. The *j-invariant* of the cross-ratio β is:

$$j(\beta) = 2^8 \frac{(\beta^2 - \beta + 1)^3}{\beta^2 (\beta - 1)^2}$$

Theorem 2.10. *$j(\beta) = j(\beta') \iff \beta, \beta'$ are cross-ratios of the same 4 points on a projective line.*

Theorem 2.11. *(The Legendre form) There is a coordinate change for the elliptic curve in the projective plane such that it assumes the form:*

$$y^2 = x(x-1)(x-\lambda)$$

with $\lambda = \frac{\alpha_3-\alpha_1}{\alpha_2-\alpha_1} \neq 0, 1$. *In this form, the three tangent lines to the curve, passing through infinity are* $x = 0, x = 1, x = \lambda$. *Moreover, as in the previous case, the fourth tangent line passing through the point at infinity is the line at infinity, which intersects the elliptic curve with multiplicity* 3.

Proof. This is given by the coordinate change:

$$\begin{cases} x = (\alpha_2 - \alpha_1)x' + \alpha_1 \\ y = \sqrt{(\alpha_2 - \alpha_1)^3} y' \end{cases}$$

□

We now want to prove the following result:

Theorem 2.12 (Salmon-1851)**.** *The* j *invariant depends only on* C *and not on the chosen marked point.*

Lemma 2.13. *Let* H_1, H_2, H_3 *and* L_1, L_2, L_3 *triples of alligned points over a cubic. Then the three lines* H_1L_1, H_2L_2, H_3L_3 *cross* C *in three distinct points* K_1, K_2, K_3, *which belong to a same line*

Proof. Follows as a direct application of the Bézout theorem. □

Proof. (Theorem)

Let $R = [0:1:0]$ and $H \in C$. The line RH meets C in a point L. The tangent line in L to C meets C in a further point L_2. Take the line RAB meeting C exactly at A and B. The lines L_2A, L_2B meet C (say) in H_2, H_3. By the lemma just stated, H, H_2, H_3 belong all to the same line. Hence we have a bijection ω between lines through R and lines through H, which turns out to be a projectivity. With ω the four tangent lines to C passing through R and H correspond each other. Hence, being j invariant under projectivity, this two four tangent lines has the same j invariant. So we changed the marked point R with an arbitrarily chosen H without changing j. □

Corollary 2.14. *Two elliptic curves are isomorphic if and only if they have the same j invariant.*

Proof. If two elliptic curves C, C' are isomorphic, we can write them in their Legendre form to find that they have the same j invariant.

Conversely, let $j(C) = j(C')$, and let C has equation:

$$y^2 = x(x-1)(x-\lambda)$$

and C' has equation:

$$y^2 = x(x-1)(x-\mu)$$

with $\mu \neq \lambda$. μ varies in the set $\{\frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\}$ Now:

- if $\mu = \lambda^{-1}$ the projectivity $\phi : x \longrightarrow \lambda x,\ y \longrightarrow \lambda^{\frac{3}{2}} y$ sends C to C';
- if $\mu = 1-\lambda$ the projectivity $\psi : x \longrightarrow -x+1,\ y \longrightarrow iy$ sends C to C';
- if $\mu = (1-\lambda)^{-1}$ the projectivity is $\phi \circ \psi$;
- if $\mu = (\lambda-1)\lambda^{-1}$ the projectivity is $\psi \circ \phi$;
- if $\mu = \lambda(\lambda-1)^{-1}$ the projectivity is $\phi \circ \psi \circ \phi$.

□

Theorem 2.15. *There is a bijection between $\mathbb{K}$ and the set of isomorphism classes of elliptic curves.*

Proof. The bijection is given associating to each isomorphism class its j module. Conversely:

- to $j = 0$ one associates the (class of isomorphism of) the curve $y^2 = x^3 + 1$;
- to $j = 1728$ one associates the (class of isomorphism of) the curve $y^2 = x^3 - x$;
- to each other j one associates the curve $y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$.

To check the assertion and all the computations, we refer to [Si] or to [Po]. □

Remark 2.16. (extension at infinity-compactification) What we have just constructed is a family of elliptic curves parametrized by the affine line $\mathbb{A}^1 - \{0, 1728\}$, with coordinate j. Let's try to extend this family with a nodal curve lying "onto" the point at infinity, hence take somehow the "limit" for $j \longrightarrow \infty$. We find out the following expression:

$$y^2 + xy = x^3$$

which represents a nodal cubic.

Observe that the only possible change of coordinates are of the form:

$$\begin{cases} x = u^2 x' \\ y = u^3 y' \end{cases}$$

with $u \neq 0$. In fact $s = t = r = 0$ because the coefficient of x^2 is 0.

Theorem 2.17. *Let C be an elliptic curve. Then its group of automorphisms is:*

- μ_2 *if* $j(C) \neq 0, 1728$*;*
- μ_4 *if* $j(C) = 1728$*;*
- μ_6 *if* $j(C) = 0$.

Proof. The curve C in the reduced Weierstrass form $y^2 = x^3 + ax + b$. Each automorphism $\sigma \in$Aut(C) has the form:

$$\begin{cases} x = u^2 x' \\ y = u^3 y' \end{cases}$$

Applying the map σ it turns out:

$$u^6 (y')^2 = u^6 (x')^3 + u^2 A x' + B$$

and dividing by u^6:

$$y'^2 = x'^3 + u^{-4} A x' + u^{-6} B$$

σ is an automorphism if and only if this is of the same form of the starting equation, i.e.:

$$\begin{cases} u^{-4} A = A \\ u^{-6} B = B \end{cases}$$

If $AB \neq 0$ then $j(C) \neq 0, 1728$ and Aut$(C) \cong \mu_2$, if $A = 0$ then $j = 0$ and Aut$(C) \cong \mu_6$, finally if $B = 0$, then $j = 1728$ and Aut$(C) \cong \mu_4$. $\square$

The following is on the field of the complex numbers. One other way to construct a genus 1 curve is to take the quotient of $\mathbb{C}$ by a lattice. Recall that a lattice in a finite dimensional real vector space V is a finitely generated subgroup Λ of V with the property that a basis of Λ as an abelian group is also a basis of V as a real vector space. A lattice in $\mathbb{C}$ is thus a subgroup Λ of $\mathbb{C}$ that is isomorphic to $\mathbb{Z}^2$ and is generated by two complex numbers that are not real multiples of each other. in this case, the quotient group $\mathbb{C}/\Lambda$ is a compact Riemann surface which is diffeomorphic to the product of two circles, and so of genus 1 (recall the equality between the three genera). Take now an elliptic curve of genus 1, our task now is to find the corresponding lattice.

Proposition 2.18. *Let C be a curve of genus 1. Then a non-zero holomorphic differential on C has no zeros.*

Proof. It is an application of the theorem of Riemann-Roch. One can also check it by hands, as done in [Du]. □

Fix now a non-zero holomorphic differential ω on C. Every other holomorphic differential is a multiple of ω since the dimension of holomorphic differentials is 1. The period lattice of C is defined to be:

$$\Lambda = \{ \int_c \omega \;\mid\; c \in H_1(C, \mathbb{Z}) \}$$

This is easily seen to be a subgroup of $\mathbb{C}$. This in fact is a lattice, if $\int_a \omega$ and $\int_b \omega$ are linearly dependent over $\mathbb{R}$ this would contradict the Hodge decomposition of $H^1(C, \mathbb{C})$.

Proposition 2.19. *Fix a base point $x_0 \in C$. Define a map $\nu : C \longrightarrow \mathbb{C}/\Lambda$ by:*

$$\nu(x) = \int_\gamma \omega$$

where γ is any smooth path in C that goes from x_0 to x. Then:

1. *ν is well defined*

2. *ν is holomorphic*

3. *ν has nowhere vanishing differential, and is therefore a covering map*

4. *the homomorphism* $\nu_* : \pi_1(C, x_0) \longrightarrow \pi_1(\mathbb{C}/\Lambda, 0)$ *is surjective, and therefore an isomorphism*

Hence ν *is a biholomorphism.*

Remark 2.20. If C is an algebraic curve of genus 1, then the automorphism group of C acts transitively on C. Consequently, the natural mapping that takes $[C; x]$ to $[C]$ is a bijection.

Proof. This follows as every genus 1 riemann surface is isomorphic to one of the form $\mathbb{C}/\Lambda$. For such Riemann surfaces, we have the homomorphism:

$$\mathbb{C}/\Lambda \longrightarrow \mathrm{Aut}(\mathbb{C}/\Lambda)$$

that takes the coset $a + \Lambda$ to the translation $z + \Lambda \mapsto z + a + \Lambda$. □

Lemma 2.21. *Let* $\mathbb{C}/\Lambda$ *and* $\mathbb{C}/\Lambda'$ *be two complex tori, and let* ω_1, ω_2*,* ω_1', ω_2' *be respectively basis for the two lattices. Each map between the two tori satisfies:*

$$\begin{cases} f(z + 2\omega_1) = f(z) + 2m_{11}\omega_1' + 2m_{12}\omega_2' \\ f(z + 2\omega_2) = f(z) + 2m_{21}\omega_1' + 2m_{22}\omega_2' \end{cases}$$

In particular, if the map has to be holomorphic, it is of the form:

$$z \mapsto az + b$$

Finally, two such complex tori are biholomorphically equivalent if and only if $\tau := \omega_2/\omega_1$ *and* $\tau' = \omega_2'/\omega_1'$ *satisfy:*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Theorem 2.22. *Fix* $\mathbb{K} = \mathbb{C}$*. Then there is a bijection:*

$$\{\textit{Elliptic Curves}/\cong\} \longrightarrow \{\textit{Complex Tori}/\cong\}$$

Proof. (sketch) Up to now we have only checked that to any elliptic curve, corresponds a complex torus. To get the inverse morphism, one has to use the Weierstrass $\mathcal{P}$ function related to the lattice Λ. This construction is given in [Ca2]. The following:

$$z \longrightarrow (\mathcal{P}(z), \mathcal{P}'(z))$$

gives a well defined map from $\mathbb{C}/\Lambda$ in $\mathbb{A}^2$. Furthermore, one can check that the image of such map satisfies:

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 + a\mathcal{P}(z) + b$$

where a, b depend on the lattice ([Ca2] and [Du] for details). One can check that such a map is an isomorphism between complex varieties.

Now fix an elliptic curve modulo isomorphism. Forgetting the marked point gives a bijection with genus 1 curves. Then the previous construction associates to such genus 1 curve a complex torus, whose associated elliptic curve is the starting one.

Conversely, starting from a complex torus, one finds out an elliptic curve. The complex torus associated to such elliptic curve could be different, because of the choice of the basis of the homology on the elliptic curve. One can easly check that the new basis for the lattice ω_1', ω_2' satisfies:

$$\begin{cases} \omega_2' = a\omega_2 + b\omega_1 \\ \omega_1' = c\omega_2 + d\omega_1 \end{cases}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. The ratio $\tau = \omega_2/\omega_1$ transforms as in the previous lemma. Hence, by the last lemma, the isomorphism class of the new complex torus is the same. $\square$

Remark 2.23. A fundamental domain for the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is the region:

$$\{\tau \in \mathbb{H} |\ \ |\mathrm{Re}(\tau)| \leq 1/2,\ |\tau| \geq 1\}$$

Observe that $\tau = i$ corresponds to the elliptic curve $y^2 = x^3 + x$ (j=1728) and the point $\tau = \frac{1+\sqrt{3}}{2}$ to the elliptic curve $y^2 = x^3 + 1$ (j=0).

Remark 2.24. The quotient $\mathbb{H}/SL_2(\mathbb{Z})$ is isomorphic, as complex 1-dimensional variety (not compact, of course) to $\mathbb{C}$. The isomorphism is given by the exponential map.

3 Moduli stacks of smooth and stable curves

The moduli space of smooth $n-$ pointed genus g curves, denoted $\mathcal{M}_{g,n}$, parametrizes isomorphism classes of objects of the form $(C; p_1, ..., p_n)$ where C is a smooth genus g

curve, and $p_1, ..., p_n$ are distinct points of C, provided that $2g-2+n > 0$. The points of $\overline{\mathcal{M}}_{g,n}$ correspond to isomorphism classes of stable $n-$pointed genus g curves. We review now the notion of dual graph of a stable curve, since it turns out to be a useful featur to think of such a curve.

Definition 3.1. The *dual graph* Γ associated to a stable curve, is a set $V = V(\Gamma)$ of *vertices* and a set $L = L(\Gamma)$ of *half-edges*. The set V is just the set of components of the normalization N of C, while L is the set of all points of N mapping to a node or to one of the p_i. The elements of L mapping to nodes come inpairs, the *edges* of the graph, while the remaining ones are called *legs*. For any $v \in V$,we let g_v be the genus of the corresponding component of N, L_v the set of half-edges incident to v and l_v its cardinality. In addition, the numbering of the p_i yields a numbering of the legs.

Hence the (arithmetic) genus of C, according to 1.18 can be read off from its graph, and is nothing but the sum of the g_v plus the number of edges minus the number of vertices plus one, or, more compactly:

$$g_c = \sum g_v + \#\text{edges} - \#\text{vertices} + 1$$

The graph is said to be *stable* if $2g_v - 2 + l_v > 0$ for any vertex v. It is immediate to see that a curve is stable if and only if its dual graph is stable.

Hence, from now on, in this subsection, we fix a scheme S of finite type over $\mathcal{K}$ and two non negative integers g, n, such that $2g - 2 + n > 0$.

Definition 3.2. A family of curves parametrized by S is a morphism of schemes $\pi : C \longrightarrow S$ such that each geometric fiber is a curve.

Definition 3.3. If $\pi : C \longrightarrow S$ is a family of curves and $f : S' \longrightarrow S$ is a morphism of schemes, there is the pullback family induced over S' by f. The fiber is canonically isomorphic.

Definition 3.4. Given two families of curves $\pi_1 : C_1 \longrightarrow S_1$ and $\pi_2 : C_2 \longrightarrow S_2$ a *morphism* F from the family π_1 to the family π_2 is a couple of morphisms $f : C_1 \longrightarrow C_2$ and $g : S_1 \longrightarrow S_2$ such that the following diagram is cartesian:

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "g"] & S_2 \end{tikzcd}$$

Remark 3.5. If the map downstairs is an iso, then also the other is an iso.

Without any property for the morphism defining the family, the families could vary too much. For projective curves one invariant which is useful not to modify in such variations is the Hilbert polynomial. The Hilbert polynomial carries geometrical informations such as the genus, the dimension and the degree, all the coefficients have a precise geometrical meaning. There is the following theorem:

Theorem 3.6 ([H] thm. 9.9 chap.3)**.** *A family of closed subscheme of a projective space over a reduced base is flat if and only if all fibers have the same Hilbert polynomial.*

In particular, for curves, flat is the same as asking that the genus and the degree are constant.

Definition 3.7. An n-pointed smooth curve of genus g over S is a flat and proper morphism $\pi : C \longrightarrow S$ together with n distinct sections $s_i : S \longrightarrow C$ such that each geometric fiber is a smooth, genus g curve. Furthermore the sections are different $s_i(x) \neq s_j(x)$ for all x geometric point of S, provided $i \neq j$. $\mathcal{M}_{g,n} : \mathcal{SCH}^{op} \longrightarrow \mathcal{SET}$ is the functor which sends each scheme T in the set of flat proper families over T of smooth curves of genus g with n marked points, up to isomorphism.

Definition 3.8. An n-pointed stable curve of genus g over S is a flat and proper morphism $\pi : C \longrightarrow S$ together with n distinct sections $s_i : S \longrightarrow C$ such that each geometric fiber is a stable, arithmetic genus g curve. Furthermore the sections are different $s_i(x) \neq s_j(x)$ for all x geometric point of S, provided $i \neq j$, and $s_i(x)$ has to be a smooth point of the overlying curve. $\overline{\mathcal{M}}_{g,n} : \mathcal{SCH}^{op} \longrightarrow \mathcal{SET}$ is the functor which sends each scheme T in the set of flat proper families over T of stable curves of genus g with n marked points, up to isomorphism.

Remark 3.9. It is usual to denote by $\overline{\mathcal{M}}^0_{g,n}$ the contravariant functor associating the set of isomorphism classes of flat families over T with automorphism group trivial.

Definition 3.10. If the functor $\mathcal{M}_{g,n}$ happens to be representable, the universal object $(M_{g,n}, \psi)$ representing it is said to be the fine moduli space for the moduli problem. The same for $\overline{\mathcal{M}}_{g,n}$.

Remark 3.11. By Yoneda's lemma, if it exists, it is unique. There is an isomorphism:

$$\psi_K : \mathcal{M}_{g,n}(SpecK) \longrightarrow h_{M_{g,n}}(SpecK) \cong M_{g,n}$$

This tells us that the geometric points are naturally in bijection with the set of isomorphism classes of curves of the type considered.

Definition 3.12. $(M_{g,n}, \psi)$ with the first a scheme and the second a natural transformation from $\mathcal{M}_{g,n}$ to $h_{M_{g,n}}$ is a coarse moduli space for the moduli problem iff:

1. the map $\psi_K : \mathcal{M}_{g,n}(SpecK) \longrightarrow h_{M_{g,n}}(SpecK)$ is a bijection
2. for every scheme N and every natural transformation $\phi : \mathcal{M}_{g,n} \Rightarrow h_N$, there is a unique morphism of functor ψ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M}_{g,n} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & h_N \\
& h_{M_{g,n}} \arrow[u, dashed]
\end{tikzcd}$$

Remark 3.13. The two conditions give the two conditions of the previous remark. A fine moduli space is coarse, but the converse is not true.

Example 3.14. $\mathbb{A}^1_{\mathbb{K}}$ is a coarse moduli space for $\mathcal{M}_{1,1}$. Let' s define a natural transformation $\psi : \mathcal{M}_{1,1} \longrightarrow h_{\mathbb{A}^1}$ as follows: to each scheme T we associate to the isomorphism class of a family $f : X \longrightarrow T$ in $\mathcal{M}_{1,1}(T)$ the map $T \longrightarrow \mathbb{A}^1$ which sends each closed point $t \in T$ in the j coordinate of $\mathbb{A}^1$ corresponding to the module of the fiber C_t over t. This gives a morphism of schemes. If $f' : X' \longrightarrow T$ is a family isomorphic to f, then $X' \cong X$, hence the fibres on same points are isomorphic, and it follows that our natural transformation ψ is well defined. Now we check that the two conditions for a coarse moduli space are satisfied:

1. We have seen in the chapter on curves of genus 1 exactly that $\psi_{\mathbb{K}} : \mathcal{M}_{1,1}(\mathrm{Spec}(\mathbb{K})) \longrightarrow h_{\mathbb{A}^1}(\mathrm{Spec}(\mathbb{K}))$ is a bijection.
2. Let N and ϕ be given as in the definition of coarse moduli space. The fibers of the family $y^2 = x^3 + ax + b$ on $\mathbb{K}^2 \setminus (0,0)$ are all the elliptic curves modulo isomorphism. The natural transformation ϕ maps such family in a morphism $f : \mathbb{K}^2 \setminus (0,0) \longrightarrow N$ which is determined by its values on points since the first

space is reduced. If $(a,b) = (\lambda^4, \lambda^6)$ for a given λ, then $f(a,b) = f(\lambda^4 a, \lambda^6 b)$. Now we are in the following situation:

$$\begin{tikzcd} \mathbb{K}^2 \smallsetminus (0,0) \arrow[r, "f"] \arrow[d, "j"] & N \\ \mathbb{A}^1 \arrow[ur, dotted, "p"'] & \end{tikzcd}$$

and we look for a morphism p making the diagram commute. On the closed points p sends a point $j \in \mathbb{A}^1$ into the image $f(a,b)$, where (a,b) are a fiber of the given family with j-invariant j. If $j(a,b) = j(a',b')$, then $(a',b') = (\lambda^4 a, \lambda^6 b)$, hence $f(a,b) = f(a',b')$ and p is well defined. From descent theory it follows that this is a morphism of schemes. This p determines the natural trasformation $\pi : h_{\mathbb{A}^1} \longrightarrow h_n$

The moduli space $\mathbb{A}^1$ is not a fine moduli space. A quick way to prove this fact follows considering the family of elliptic curves given by:

$$\lambda y^2 = x^3 + x + 1$$

is a family of elliptic curves which are all isomorphic to say $E: \; y^2 = x^3 + x + 1$, but they have only finitely many sections. In fact, a section corresponds to a map $g : \mathbb{A}^1 \smallsetminus 0 \longrightarrow E$ satisfying $g(-\lambda) = -g(\lambda)$, so sections correspond to $2-$torsion points of E which are four. But if the family were trivial, it would have infinitely many sections (obviously). Let's now check explicitly that our coarse moduli space is not a fine moduli space. We consider the family $f : C \longrightarrow \mathbb{A}^1 \smallsetminus 0$ with $C \subset \mathbb{P}^2 \times (\mathbb{A}^1 \smallsetminus 0)$ whose fibers C_t have equation $y^2 - x^3 - t = 0$. All curves C_t has the same j-invariant $j = 0$, but the family is not trivial. If the family were trivial, there would exist an isomorphism $\phi_t : C_t \longrightarrow C_1$. Such ϕ_t has to be of the form $\phi_t(x) = u_t^2 x$, $\phi_t(y) = u_t^3 y$, where $u_t^6 = t$. But a regular morphism $u : \mathbb{A}^1 \smallsetminus 0 \longrightarrow \mathbb{A}^1$ such that $u^6(t) = t$ cannot exist. In an analogous way, one can check that $\mathbb{P}^1_{\mathbb{K}}$ is a coarse moduli space for $\overline{\mathcal{M}}_{1,1}$ which is not a fine moduli space.

Example 3.15. The coarse moduli space doesn't always exist. Let F be the functor associating to each scheme T the set of isomorphism classes of flat family of reduced plane conic over $\mathbb{K}$. $F(\mathrm{Spec}\mathbb{K})$ contains two elements: the smooth irreducible conic and the couple of distinct lines. We have seen that the second condition of being a coarse moduli space determines it up to isomorphism. There exists a simple natural transformation $\psi : F \longrightarrow h_{pt}$ which is given by the constant morphism $T \longrightarrow pt$. The couple (pt, ψ) satisfies the second condition of the definition of coarse moduli space, without satisfying the first ($|F(pt)| = 2$, on the contrary $|h_{pt}(pt)| = 1$). So F can't have a coarse moduli space.

Here there is a simple example.

Theorem 3.16. *([Kd]) For $n \geq 3$, $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ are representable functors. Their respectively representing schemes are quasiprojective (projective) smooth irreducible varieties of dimension $n - 3$.*

Theorem 3.17. *([DM] M^0 are sheaves in the Zariski topology, so they are representable.*

Here we give a particular example, which turns out to be really useful in the last chapter:

Theorem 3.18. *$\mathcal{M}_{1,n}$ is a representable functor if $n \geq 5$.*

Proof. Follows from the fact that a smooth elliptic curve with 5 marked points has no automorphisms. □

Remark 3.19. From now on our moduli functors, will be seen as pseudofunctors in groupoids.

Theorem 3.20. *[DM] $\mathcal{M}_{g,n}$ is a stack in the étale topology, and hence in the Zariski topology and so is $\overline{\mathcal{M}}_{g,n}$.*

Remark 3.21. Up to now we have two notions of *coarse*: the coarse space associated to a stack, and the coarse moduli space of a moduli problem. When the given stack arises from a moduli problem, the two notions coincide.

Theorem 3.22 ([DM])**.** *$\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$ are Deligne-Mumford algebraic stacks, smooth irreducible of dimension $3g - 3 + n$. The first is proper and the second is open dense in the first.*

Remark 3.23. The whole construction of our new spaces: algebraic stacks as pseudofunctor in groupoids, has as pourpose to find a fine moduli space instead of just a coarse one. One in fact has, tautologically, that $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ as algebraic stacks are fine moduli spaces.

Now, according to [Co1], there are basically four algebraic proofs of the projectivity of the moduli spaces of stable curves: the original one by Knudsen [Kd], the proof by Mumford-Giesker and a more recent by Viehweg and Kollár. The last one is given in the short Cornalba's paper [Co1]:

Theorem 3.24. *The coarse moduli space associated to $\overline{\mathcal{M}}_{g,n}$ is a reduced algebraic space which is proper and separated. Actually, it is a projective scheme. Moreover, the coarse moduli space associated to $\mathcal{M}_{g,n}$ is a reduced algebraic space which is separated, and it is a quasiprojective scheme.*

Theorem 3.25. *(Cohomology and Base Change) [[H] chap.3, Thm 12.11] Let $f : X \longrightarrow Y$ be a projective morphism of schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Let y be a point of Y. Then*

1. *if the natural map:*
$$\phi^i(y) : R^i f_*(\mathcal{F}) \otimes \mathbb{K}(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$
is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighbourood of y.
2. *Assume that $\phi^i(y)$ is surjective. Then the following conditions are equivalent:*
   - *$\phi^{i-1}(y)$ is also surjective.*
   - *$R^i f_*(\mathcal{F})$ is locally free in a neighbourood of y.*

Theorem 3.26. *$\overline{\mathcal{M}}_{1,1}$ is a stack. It is isomorphic to the global quotient stack: $[\mathbb{C}^2 \smallsetminus (0,0)/\mathbb{C}^*]$ where $\mathbb{C}^*$ acts (as usual) as:*

$$\lambda(x) := \lambda^4 x \quad \lambda(y) := \lambda^6 y$$

Proof. We have a natural morphism from $[\mathbb{C}^2 \smallsetminus (0,0)/\mathbb{C}^*]$ onto $\overline{\mathcal{M}}_{1,1}$, since onto the first space we have a tautological family of elliptic curves (with parameters (a, b)). We first observe that the results of the section about elliptic curves guarantees that the natural morphism is an isomorphism when we compute all (stacks and morphism) on a geometric point. In fact, every elliptic curve is, up to isomorphism, a curve of the form $y^2 = x^3 + ax + b$ (essential surjectivity). Moreover, since we proved that every automorphism of an elliptic curve is of the form λ, where λ acts as usual on (x, y, z, a, b), the functor between the two groupoids is fully faithful.

Now we have to prove a similar result when all is computed onto a generic scheme S. In the case with one point, we took for the Weierstrass embedding theorem $L = \mathcal{O}(p)$, then we had $\chi(L)^{\otimes n} = n$ thanks to RR. Furthermore, since the curve is elliptic, we had $H^1(L^{\otimes n})^v = H^0(L^{\otimes -n}) = 0$, and so the dimension of the global sections of the n^{th}-power of L is exactly n. From this we found out the equations.

In the case when all is computed over a generic scheme S, one has a family:

$$\begin{tikzcd} E_S \arrow[d, "f"] \\ S \arrow[u, bend left, "s"] \end{tikzcd}$$

One than takes as $L := \mathcal{O}_{E_S}(sS)$, observing that L restricted to a geometric point of S is the same L as before. Now we apply 3.25 with $X = E_S$, $Y = S$ and $\mathcal{F} = L^{\otimes n}$. The first point with $i = 1$ tells that, being $H^1(E_y, \mathcal{F}|_{E_y}) = 0$, $\mathrm{R}^1 f_* \mathcal{F} \otimes \mathbb{K}(y') = 0 \forall\, y' \in U(y)$, where $U(y)$ is a suitably chosen Zariski neighbourood of y. Nakayama applied to the left hand side of the tensor product guarantees that $\mathrm{R}^1 f_* \mathcal{F} = 0$. Now using the second point, we have that $\phi^0(y)$ is surjective, hence again by the first point an isomorphism. Now $\mathrm{R}^0 f_* \mathcal{F}$ is just $f_* \mathcal{F}$. Using the second point once again, one finds out that, restricting the neighbourood the sheaf $f_* \mathcal{F}$ is locally free.

Now one can argue as in the Weierestrass embedding theorem, covering the scheme S with open subsets trivializing the fiber bundle, and then embedding families of elliptic curves in the projective space instead of single elliptic curves. □

4 $\overline{\mathcal{M}}_{g,n}$ as universal curve over $\overline{\mathcal{M}}_{g,n-1}$

The standard reference for this section is [Kd]. For a more informal descripion, see [L] and [Co2].

Definition 4.1. (contraction) The following functor:

$$\overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

Definition 4.2. (clutching) The following two functors:

$$\mathcal{M}_{g,n+2} \longrightarrow \mathcal{M}_{g+1,n}$$

$$\mathcal{M}_{g,n_1+1} \times \mathcal{M}_{g,n_2+1} \longrightarrow \mathcal{M}_{g_1+g_2,n_1+n_2}$$

Theorem 4.3 (Knudsen)**.** *The contraction functor is representable and moreover it is isomorphic to the universal curve over the underlying space.*

Proof. (sketch) Let $(C; x_1, ..., x_n)$ be a stable curve of type (g, n). Let's show that any $x \in C$ determines a stable curve $(\tilde{C}; \tilde{x}_1, ..., \tilde{x}_{n+1})$ of type $(g, n+1)$:

- If x is a regular point of C and it is not one of the x_i's, then take $(\tilde{C}; \tilde{x}_1, ..., \tilde{x}_{n+1}) = (C; x_1, ..., x_n, x)$.
- If $x = x_i$ for some i, we let $\tilde{C}$ be the disjoint union of C and $\mathbb{P}^1$ with the points x_i and ∞ identified. We let $\tilde{x}_i = 1 \in \mathbb{P}^1$ and $\tilde{x}_{n+1} = 0 \in \mathbb{P}^1$ whereas for $j \neq i, n+1, \tilde{x}_j = x_j$, viewed as a point of $\tilde{C}$ in the obvious way. We denote this $(n+1)$-pointed curve by $\sigma_i(C; x_1, ..., x_n)$.
- If x is a singular point of C, then $\tilde{C}$ is obtained normalizing C in this point only and by putting back a copy of $\mathbb{P}^1$ with $\{0, \infty\}$ identified with the preimage of x. Then $\tilde{x}_{n+1} = 1 \in \mathbb{P}^1$ and for $i \leq n,\ \tilde{x}_i = x_i$, viewed as a point of $\tilde{C}$ in the obvious way. We thus have defined a map $C \longrightarrow \overline{\mathcal{M}}_{g,n+1}$ that maps x_i to σ_i.

The converse construction associates to a $(n+1)$ pointed stable curve the stable curve n pointed given basically forgetting the last marked point. This yields a stable pointed curve unless this last point lies on a smooth rational component which has only two other special points. Let $\tilde{C}$ be obtained by contracting this component and let x_i be the image of $\tilde{x}_i,\ i \leq n$. Notice that the map $C \longrightarrow \overline{\mathcal{M}}_{g,n+1}$ defined above parametrizes the fiber of π over the point defined by $(C; x_1, ..., x_n)$. One now has to check that the given map π is a morphism and so are its sections σ_i. Moreover the fiber of π over the point defined by $(C, x_1, ..., x_n)$ can be identified with the quotient of C by the automorphisms of the pointed curve. [Kd].

□

Theorem 4.4. *The functor neglecting last $n-h$ points:*

$$\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,h}$$

is representable.

Proof. By induction on $n-h$, since the composite of representable morphisms is representable, it is sufficient to work out the case $n-h=1$. Representability is stable under base change and $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open embedding, which is representable by definition. So the functor neglecting the last marked point is representable also when dealing with smooth curves, and the statement follows. □

Definition 4.5. The tautological universal family of curves over $\mathcal{M}_{g,n}$ will be called $C_{g,n}$, over $\overline{\mathcal{M}}_{g,n}$ will be called $\overline{C}_{g,n}$.

Theorem 4.6 (Lemma 4.4.3 [AV2], Lemma 3.3.2 [C])**.** *Let $g : \mathcal{G} \longrightarrow \mathcal{F}$ be a morphism of Deligne Mumford stacks. The following conditions are equivalent:*

- *The morphism $g : \mathcal{G} \longrightarrow \mathcal{F}$ is representable.*
- *For any $\xi \in \mathcal{G}(k)$, the natural group homomorphism $Aut(\xi) \longrightarrow Aut(g(\xi))$ is injective.*

Appendix D

Orbifold Cohomology for $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$

1 The dimension: description of twisted sectors

If A is a commutative ring, we have defined $H^*_{orb}(F, A) = H^*(I_F, A)$ as $A-$modules. If F_1 and F_2 are different connected components of F, $H^*(F_1 \coprod F_2)$ and $H^*(F_1) \bigoplus H^* F_2$ are isomorphic as A-modules. We will take rational coefficients, hence by 2.7 we are only interested in the coarse moduli space associated to the inertia stack. So, since we start our computation of $H^*_{orb}(\overline{\mathcal{M}}_{1,n})$ by understanding it as a vector space, the first step is to write down all their connected components.

To simplify the notation, we will not write any marked point when describing the connected components of the inertia stack. Anyway, marked points will be clear from the context or explicitly described in the proofs.

Proposition 1.1. $I(\mathcal{M}_{1,1}) = (\mathcal{M}_{1,1}, 1) \coprod (\mathcal{M}_{1,1}, -1) \coprod (y^2 = x^3 + x, i) \coprod (y^2 = x^3 + x, -i) \coprod (y^2 = x^3 + 1, \epsilon) \coprod (y^2 = x^3 + 1, \epsilon^2) \coprod (y^2 = x^3 + 1, \epsilon^4) \coprod (y^2 = x^3 + 1, \epsilon^5)$

Proposition 1.2. $I(\overline{\mathcal{M}}_{1,1}) = (\overline{\mathcal{M}}_{1,1}, 1) \coprod (\overline{\mathcal{M}}_{1,1}, -1) \coprod (y^2 = x^3 + x, i) \coprod (y^2 = x^3 + x, -i) \coprod (y^2 = x^3 + 1, \epsilon) \coprod (y^2 = x^3 + 1, \epsilon^2) \coprod (y^2 = x^3 + 1, \epsilon^4) \coprod (y^2 = x^3 + 1, \epsilon^5)$

Proof. We have denoted in both statements a point inside the moduli space with its equation as a projective curve (via Weierstrass representation). The statement is then a consequence of 2.17. One checks that each component is open inside the inertia stack, and its complement is open too, moreover the given decomposition is the decomposition connected components. □

One first attempt (just looking at the definition of the inertia stack) is to go on by induction on the number of marked points trying to prove that the following:

$$
\begin{tikzcd}
 & I(\overline{M}_{1,1}) \arrow[rr] \arrow[dd] & & \overline{M}_{1,1} \arrow[dd] \\
I(\overline{M}_{1,2}) \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & \overline{M}_{1,2} \arrow[ur] & \\
 & \overline{M}_{1,1} \arrow[rr] & & \overline{M}_{1,1} \times \overline{M}_{1,1} \\
\overline{M}_{1,2} \arrow[ur] \arrow[rr] & & \overline{M}_{1,2} \times \overline{M}_{1,2} \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

is a 2-cartesian cube, i.e. a cube whose faces are all 2-cartesian. Unfortunately, only the front and the back side of the cube are 2-cartesian. The right and left side of the cube are not 2-cartesian, since the fiber product $(X \times X) \times_{Y \times Y} Y$ is in general $X \times_Y X$ which is different from X (in a general category).

A less categorical and more geometrically-illuminating observation, is that the two remaining sides of the cube are not 2-cartesian since the automorphism group of a 2-marked curve is, in general, strictly smaller than the automorphism group of the corresponding 1-marked curve.

Taking into account this last consideration, we proceed for the computation of the dimension for H^*_{orb} by induction on the number of marked points, using the fact that $\overline{\mathcal{M}}_{1,n+1} \longrightarrow \overline{\mathcal{M}}_{1,n}$ is the universal curve, and using the representability of the functor which forgets points. Every time there is an extra marked point given, every twisted sector somehow "splits up". From now on, the elliptic curve $y^2 = x^3 + x$ will be called C_4 and $y^2 = x^3 + 1$ will be called C_6, in order to remember their peculiarity, i.e. their nontrivial stabilizers being respectively μ_4 and μ_6. The inertia stack can be written more compactly:

$I(\overline{\mathcal{M}}_{1,1}) = (\overline{\mathcal{M}}_{1,1}, 1) \coprod (\overline{\mathcal{M}}_{1,1}, -1) \coprod (C_4, i/-i) \coprod (C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5)$

where "/" means that more than one automorphism is involved, and so there are isomorphic copies of twisted sectors, up to a change of the automorphism. We start with the simpler sector, the untwisted one:

Remark 1.3. (See 3.6) The inertia stack of $\overline{\mathcal{M}}_{1,n}$ has a component (the untwisted one) which is $(\overline{\mathcal{M}}_{1,n}, Id)$, as well as the inertia stack of $\mathcal{M}_{1,n}$ has a component which is $(\mathcal{M}_{1,n}, Id)$.

As stated before, let's study what happens to each twisted sector, when removing a marked point: we saw that the morphism which forgets one marked point gives an injection of the stabilizer groups, as we observed in 4.3 4.4 4.6.

1.a The case $\mathcal{M}_{1,n}$

Let's look at first at the fibers of the isolated points of the inertia stack.

Lemma 1.4. *$I(\mathcal{M}_{1,n})$ has no twisted sectors with automorphisms different from -1 if $n > 3$.*

$I(\mathcal{M}_{1,2})$ contains the following twisted sectors:

$$(C_4, i) \coprod (C_4, -i) \coprod (C_6, \epsilon^2/\epsilon^4)$$

$I(\mathcal{M}_{1,3})$ contains the following twisted sectors:

$$(C_6, \epsilon^2/\epsilon^4)$$

No other twisted sector arise from the fiber of the isolated points.

Proof. Studying the fixed points for the action on the curves C_4 and C_6 of the two groups μ_4 and μ_6, it turns out that

- i and $-i$ act on C_4 with $(0,0)$ as only fixed point different from infinity.
- ϵ, ϵ^5 act on C_6 with no fixed points different from infinity.
- ϵ^2, ϵ^4 act on C_6 with two fixed points: $(0,1)$ and $(0,-1)$. Recall that the automorphisms i and $-i$ exchange the two C_6 curves with the two different possible marked points. Hence in the moduli space, this component is to be counted only once. These last remarks conclude the lemma.

□

In the following, we study the fiber in the twisted sector $(\mathcal{M}_{1,n}, -1)$ w.r.t. the morphism which forgets the last marked point.

If S is a scheme, a morphism $\phi : S \longrightarrow \mathcal{M}_{1,1}$ is the same as a family of smooth genus 1 curves with a section s:

$$\begin{tikzcd} E_S \arrow[d, shift right] \\ S \arrow[u, bend right, "s"'] \end{tikzcd}$$

with a section s.

Definition 1.5. We define the following pseudofunctor in groupoids A_1. If S is a scheme, $A_1(S)$ is the following collection of data:

$$\begin{tikzcd} E_S \arrow[d] \\ S \arrow[u, bend left, "t"] \arrow[u, bend right, "s"'] \end{tikzcd}$$

Such that for all $u \in S$(geometric point), $t(u) \neq s(u)$, and

$$\begin{tikzcd} E_S \arrow[d, shift right] \\ S \arrow[u, bend right, "s"'] \end{tikzcd}$$

Is a map into $\mathcal{M}_{1,1}$. Moreover, we ask the section t to belong always to the set of $2-$torsion points of the elliptic curve, once the identity element is fixed by s. In a similar way, we define A_2 as the pseudofunctor associating to a scheme S the previous diagram but with three sections s, t_1, t_2 (different at all points and all belonging to the two-torsion set). Finally, we define A_3 as the pseudofunctor associating to a scheme S the previous diagram with four sections s, t_1, t_2, t_3 in the same way (now all the two torsion points have been chosen).

In the following, we will call U the following subset of the affine plane:

$$U := \{(a, b)| \ \ 4a^3 - 27b^2 \neq 0\} \subset \mathbb{A}^2$$

Lemma 1.6. *The pseudofunctor A_1 previously defined is isomorphic to $[W/\mathbb{C}^*]$, where:*

$$W = \{(x, y, z, a, b)| \ \ zy^2 = x^3 + axz^2 + bz^3, \ \ y = 0\} \subset \mathbb{P}^2 \times U$$

and $\mathbb{C}^*$ *acts as usual as:*

$$x \longrightarrow \lambda^2 x \quad y \longrightarrow \lambda^3 y \quad z \longrightarrow z$$

$$a \longrightarrow \lambda^4 a \quad b \longrightarrow \lambda^6 b$$

$\mathbb{C}^*$ *acts with finite stabilizers. Hence* A_1 *is an algebraic stack which is a global quotient.*

Proof. We have a morphism from $[W/\mathbb{C}^*]$ to A_1, since the first space has a tautological family of elliptic curves (given by $y^2 = x^3 + ax + b$, the first marked point is at infinity), with a second section, which, over a point (a, b, x) such that $(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = 0$ is exactly the corresponding α_i. Now the embedding of Weierstrass again guarantees that, when computed onto geometric points, the resulting groupoids are isomorphic via the given morphism. To get the same result for families, apply an argument of Cohomology and Base Change (3.25) as in 3.26. $\square$

Let's now call:

$$V = \{y^2 = x^3 + axz^2 + bz^3\} \subset U \times \mathbb{P}^2$$

$$W_2 := W \times_U W \smallsetminus \text{diagonal} =$$

$$= \{(a, b, x_i, y_i, z_i)| \;\; z_i y_i^2 = x_i^3 + a x_i z_i^2 + b z_i^3 \;\; y_i = 0 \; i = 1, 2\} \smallsetminus \text{diagonal}$$

and

$$W_3 := W \times_U W \times_U W \smallsetminus \text{big diagonal}$$

Observe that $W_2 \sim W_3$ canonically. We state also the following proposition, whose proof we omit since it is analogous to the previous one *mutatis mutandis.*

Proposition 1.7. A_2 *is isomorphic to* $[W_2/\mathbb{C}^*]$*, and* A_3 *is isomorphic to* $[W_3/\mathbb{C}^*]$*.* $\mathbb{C}^*$ *acts on both with finite stabilizers. Hence both are algebraic stacks and global quotients. So since the action is the same and the starting schemes are canonically isomorphic, the two stacks* A_2 *and* A_3 *are isomorphic. Moreover* $C_{1,1}$ *is isomorphic to* $[V/\mathbb{C}^*]$*.*

Remark 1.8. Computing all such pseudofunctors on one geometric point Spec($\mathbb{K}$), one finds out that the objects of A_1 are:

$$\{(E, 0, \eta) \mid (E, 0) \text{ ellpitic curve}, \eta \neq 0 \text{ 2-torsion point different from } 0\}.$$

Henceforth, the objects of A_2 are:

$$\{(E, 0, \eta_1, \eta_2) \mid (E, 0) \text{ ellpitic curve}, \eta_1, \eta_2, 0 \text{ different 2-torsion points}\}.$$

Finally, the objects of A_3 are:

$$\{(E, 0, \eta_1, \eta_2, \eta_3) \mid (E, 0) \text{ ellpitic curve}, \eta_1, \eta_2, \eta_3, 0 \text{ different 2-torsion points}\}.$$

These are usually called *level-2 structures* of the moduli space of the elliptic curves (see e.g. [Sc]).

Definition 1.9. Let X and Y be two orbifolds. A surjective morphism $f : X \longrightarrow Y$ will be called an *orbifold covering* when the morphism is representable, proper and étale. This is the same thanks to 2.12 as asking the morphism to be representable, finite and unramified.

Remark 1.10. This definition includes only finite coverings. Furthermore, representability is not really needed, since properness can be defined for arbitrary morphisms of algebraic stacks, and thanks to 2.12.

Here we will use the Lemma 5.5.

Lemma 1.11. *The following is a diagram of finite coverings:*

$$\begin{array}{c} A_3 \\ \downarrow 1:1 \\ A_2 \\ \downarrow 2:1 \\ A_1 \\ \downarrow 3:1 \\ \mathcal{M}_{1,1} \end{array}$$

where $A_2 \longrightarrow \mathcal{M}_{1,1}$ is a covering with group μ_3, $A_1 \longrightarrow \mathcal{M}_{1,1}$ is a covering with group S_3 (the group of permutation of a set with three elements).

Proof. Let's work out the case of A_1. We take U the smooth atlas:

$$U := \{(a, b) \in \mathbb{C}^2 \smallsetminus (0, 0) |\ 4a^3 - 27b^2 \neq 0\}$$

$$V = \{y^2 = x^3 + axz^2 + bz^3\} \subset U \times \mathbb{P}^2$$

$$W = \{(a, b, x, y, z) \in V \mid y = 0\}$$

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & A_1 \arrow[d] \\
V \arrow[r] \arrow[d] & \mathcal{M}_{1,2} \arrow[d] \\
U \arrow[r] & \mathcal{M}_{1,1}
\end{tikzcd}$$

Now we use that $\mathcal{M}_{1,2}$ is representable over $\mathcal{M}_{1,1}$ (Thm. 4.4), so V is a scheme. The first condition to get that $A_1 \longrightarrow \mathcal{M}_{1,1}$ is a finite covering is the representability.

This is guaranteed by lemma 5.5, recall $\mathcal{M}_{1,1} = [U/\mathbb{C}^*]$, $C_{1,1} = [V/\mathbb{C}^*]$ and $A_1 = [W/\mathbb{C}^*]$, by Proposition 1.6.

Now we have:

$$\begin{tikzcd}
W \arrow[r, "\text{cl.emb.}"'] & V \arrow[r, "\text{cl.emb.}"'] & U \times \mathbb{P}^2 \arrow[d] \\
 & & U
\end{tikzcd}$$

The mapping onto U is proper by completeness of the projective space. Now the scheme W is actually a smooth manifold, in fact dishomogeneizing w.r.t. the variable z, one finds out (renaming $x := x/z$):

$$W = \{(a, b, x) \mid \; x^3 + ax + b = 0, \quad \Delta = 4a^3 + 27b^2 \neq 0\} \subset \mathbb{A}^3$$

Which is a smooth manifold, being an open subset of an hypersurface given by $f = 0$ where ∇f is never 0. The morphisms are checked to be étale: in fact, being the two varieties involved smooth, this is the same as a local diffeomorphism. Now one checks easily that the projections onto the first two factors have differential not injective if and only if $3x^2 + a = 0$, and this, on our hypersurface, holds if and only if $\Delta = 0$, hence out of our chosen open subset. □

Corollary 1.12. *Each stack of the previous diagram can be included in its compact-*

ification:

$$\begin{tikzcd}
A_3 \arrow[r, "incl"] \arrow[d, "1:1\sim"'] & \overline{A_3} \arrow[d, "\sim"] \\
A_2 \arrow[r, "incl"] \arrow[d, "2:1"'] & \overline{A_2} \arrow[d] \\
A_1 \arrow[r, "incl"] \arrow[d, "3:1"'] & \overline{A_1} \arrow[d] \\
\mathcal{M}_{1,1} \arrow[r, "incl"] & \overline{\mathcal{M}}_{1,1}
\end{tikzcd}$$

The morphisms on the left extend to finite morphisms on the compactifications.

The atlases for $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$ are respectively:

$$\overline{W} = \{y^2 = x^3 + axz^2 + bz^3\} \subset \mathbb{C}^2 \smallsetminus (0,0) \times \mathbb{P}^2$$

$$\overline{W_2} := \overline{W} \times_{\mathbb{C}^2 \smallsetminus (0,0)} \overline{W} \smallsetminus \text{diagonal} =$$

$$= \{(a, b, x_i, y_i, z_i)| \;\; z_i y_i^2 = x_i^3 + ax_i z_i^2 + bz_i^3 \;\; y_i = 0 \; i = 1, 2\} \smallsetminus \text{diagonal}$$

and

$$\overline{W_3} := \overline{W} \times_{\mathbb{C}^2 \smallsetminus (0,0)} \overline{W} \times_{\mathbb{C}^2 \smallsetminus (0,0)} \overline{W} \smallsetminus \text{big diagonal}$$

They contain respectively $2, 3, 3$ points in addiction to the points of their uncompactified corrispondent. These are the points over (a, b) such that $4a^3 + 27b^2 = 0$.

Since here we are interested in orbifold cohomology without torsion (the coefficients are taken in the field of rational numbers), the cohomology of the twisted sectors can be taken on their coarse moduli spaces.

Proposition 1.13. *In the analytic category, the coarse moduli space for A_2 is $\mathbb{H}/\Gamma[2]$, the coarse moduli space for A_1 is $\mathbb{H}/\Gamma_0[2]$*

Proof. This is completely analogous to Theorem 2.22. As a reference also for this statement, one can take [Sc]. □

Lemma 1.14 ([Sc], Chapter 4)**.** *The genus of the fundamental region for $\Gamma[2]$ and for $\Gamma_0[2]$ is zero, hence it is isomorphic to a $\mathbb{P}^1$ minus a finite number of points. Its compactification is $\mathbb{P}^1$.*

The following confirms our previous result (in the analytic category):

Lemma 1.15 ([Sc], chapter 4). *The number of points added in the compactification is* 2 *for* $\mathbb{H}/\Gamma_0[2]$ *and* 3 *for* $\mathbb{H}/\Gamma[2]$.

Proof. (sketch) This is geometrically quite clear. Calling x_1, x_2, x_3 the three $2-$torsion points (different from infinity), we see that on the nodal curve the two points x_2 and x_3 become one single point (with a nodal singularity). Hence a choice of the points of $2-$torsion is a choice between two points, and a choice of a couple of points of $2-$torsion is a choice between $(x_1, x_2), (x_2, x_1), (x_2, x_3)$. □

Remark 1.16. Now let's look at the morphism onto the coarse moduli spaces, and check with the Riemann-Hurwitz formula that it is all right. The coarse moduli spaces of $\overline{A_1}$, $\overline{A_2}$ and $\overline{A_3}$, are copies of $\mathbb{P}^1$. In the following table we compute the cardinality of the fibers over the different points ∞ (the nodal curve), C_4, C_6 and a generic point p (a point different from all the previous ones).

	∞	C_4	C_6	p
$\overline{A_1}$	2	2	1	3
$\overline{A_2}$	3	3	2	6

The computation of Riemann-Hurwitz for the morphism of coarse moduli spaces associated to $\overline{A_1} \longrightarrow \overline{\mathcal{M}}_{1,1}$ brings to:

$$0 - 2 = 3(0 - 2) + \sum \text{ramification}$$

and one sees from the previous table that the degree of the ramification divisor is 4. as for the morphism of coarse moduli space associated to $\overline{A_2} \longrightarrow \overline{\mathcal{M}}_{1,1}$ one finds out:

$$0 - 2 = 6(0 - 2) + \sum \text{ramification}$$

and the degree of the ramificaton divisor is easily seen to be 10, according to the previous table.

What we observed, leads naturally to the following result:

Theorem 1.17. *The connected component of the inertia stack with the automorphism* (-1) *for* $\mathcal{M}_{1,n}$ *is:*

1. A_1 *if* $n = 2$.

2. A_2 *if* $n = 3$.

3. A_3 *if* $n = 4$.

4. *empty if* $n > 4$.

Let's now summarize all in the following:

Corollary 1.18. *The following are the coarse moduli spaces of the inertia stacks of moduli spaces of smooth elliptic curves with marked points:*

- *(The inertia stack of* $\mathcal{M}_{1,2}$*)*
$$I(\mathcal{M}_{1,2}) = (\mathcal{M}_{1,2}, 1) \coprod (A_1, -1) \coprod (C_4, i/-i) \coprod (C_6, \epsilon^2/\epsilon^4)$$
- *(The inertia stack of* $\mathcal{M}_{1,3}$*)*
$$I(\mathcal{M}_{1,3}) = (\mathcal{M}_{1,3}, 1) \coprod (A_2, -1) \coprod (C_6, \epsilon^2/\epsilon^4)$$
- *(The inertia stack of* $\mathcal{M}_{1,4}$*)*
$$I(\mathcal{M}_{1,4}) = (\mathcal{M}_{1,4}, 1) \coprod (A_3, -1)$$
- *The inertia stack of* $\mathcal{M}_{1,n}$ *for* $n > 4$ *has no twisted sectors, according to considerations in this chapter and to 3.18.*

Let's now give the results on cohomology:

Theorem 1.19. *The dimension of the orbifold cohomology of* $\mathcal{M}_{1,n}$ *with rational coefficients is:*

- 1 *if* $n = 1$;
- *dim*$(H^*(\mathcal{M}_{1,2}, \mathbb{Q})) + 6$ *if* $n = 2$;
- *dim*$(H^*(\mathcal{M}_{1,3}, \mathbb{Q})) + 5$ *if* $n = 3$;
- *dim*$(H^*(\mathcal{M}_{1,4}, \mathbb{Q})) + 3$ *if* $n = 4$;
- *dim*$(H^*(\mathcal{M}_{1,n}, \mathbb{Q}))$ *if* $n \geq 5$.

To conclude the section, recall that $H^*(\mathcal{M}_{1,n}, \mathbb{Q})$ is completely known [Co3].

1.b The case $\overline{\mathcal{M}}_{1,n}$

The following is a series of lemmas giving, one by one, the connected components of the inertia stack. This will allow us to compute cohomology, at least as a vector space. We proceed following the identical way of the previous chapter. As in the previous chapter, we avoid writing down the marked points when we write the components of the inertia stack, anyway they should be clear from the context, or described explicitly in the proofs.

In the following we use the notation $\overline{\mathcal{M}}_{0,n} \sim$ a point, for $n \leq 2$. It is an abuse of language, since by our previous definition it was the empty set. Nevertheless, this will simplify the description of the twisted sectors for $\overline{\mathcal{M}}_{1,n}$.

Moreover, in order to simplify the notation, if n is a natural number, S is an algebraic stack, we will denote by nS, n isomorphic copies of S.

We will use the standard notation for the multinomial coefficients:

$$(n_1, ..., n_k)! = \frac{(n_1 + ... + n_k)!}{n_1!...n_k!}$$

is the number of distinct way of decomposing the set $\{1, ..., n_1 + n_2 + ... + n_k\}$ in k distinct, ordered subsets $(S_1, ..., S_k)$ in such a way that n_1 of them belong to S_1,...,n_k of them belong to S_k.

Lemma 1.20 (A). *The components of the inertia stack of $\overline{\mathcal{M}}_{1,n}$ endowed with the automorphisms i and $-i$ are, up to isomorphism:*

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1 + 1, \alpha_2)!\ \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2} \coprod \overline{\mathcal{M}}_{0,n+1}$$

which lie in the fiber over C_4 when considering the morphism to $\overline{\mathcal{M}}_{1,1}$ which forgets sections.

Proof. The first two components of this kind come from adding a marked point to the twisted sector $(C_4, i/-i)$. Let's prove this fact, showing how to perform the induction. A marked point has to be invariant under automorphisms, so, in order to preserve the automorphisms $(i, -i)$ the only two possibilities are to mark the point $(0, 0)$ and to mark the point at infinity. One arrow left-down means to add the successive marked point onto the origin, one arrow right-down means to add the successive marked point onto the point at infinity.

$$\begin{tikzcd}
 & & & C_4 \arrow[dl] \arrow[dr] & & & \\
 & & C_4 \arrow[dl] \arrow[dr] & & C_4 \arrow[dl] \arrow[dr] & & \\
 & C_4 \arrow[dl] \arrow[dr] & & C_4 \arrow[dl] \arrow[dr] & & C_4 \times \overline{\mathcal{M}}_{0,4} \arrow[dl] \arrow[dr] & \\
C_4 \times \overline{\mathcal{M}}_{0,4} & & C_4 & & C_4 \times \overline{\mathcal{M}}_{0,4} & & C_4 \times \overline{\mathcal{M}}_{0,5}
\end{tikzcd}$$

Each vertex of the tree has to be counted with multiplicity equal to the number of paths to reach it from the upper vertex. Since C_4 inside the moduli space is a point, it is possible to omit it in the product up to isomorphism. In the following graph with (a, b) is meant the product $\overline{\mathcal{M}}_{0,a} \times \overline{\mathcal{M}}_{0,b}$. Remember that here with the moduli space of genus zero curves with less or equal than three marked points we mean a point, according to our previous convention:

$$\begin{tikzcd}
 & & & & (1,2) \arrow[dl] \arrow[dr] & & & & \\
 & & & (2,2) \arrow[dl] \arrow[dr] & & (1,3) \arrow[dl] \arrow[dr] & & & \\
 & & (3,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & & (1,4) \arrow[dl] \arrow[dr] & & \\
 & (4,2) \arrow[dl] \arrow[dr] & & (3,3) \arrow[dl] \arrow[dr] & & (2,4) \arrow[dl] \arrow[dr] & & (1,5) \arrow[dl] \arrow[dr] & \\
(5,2) & & (4,3) & & (3,4) & & (2,4) & & (1,6)
\end{tikzcd}$$

The multiplicative coefficient arises from the number of way of decomposing $\{2, ..., n\}$ in two disjoint ordered subsets. The number of points marked onto 0 is $\alpha_1 + 1$, the number of points marked onto ∞ is $\alpha_2 + 1$. $\square$

Let's now look at the two more components of the twisted sectors over C_6.

Lemma 1.21 (B).

The components of the inertia stack of $\overline{\mathcal{M}}_{1,n}$ when $n \geq 2$ endowed with the auto-

morphisms ϵ^2 and ϵ^4 are, up to isomorphism:

$$\coprod_{\alpha_1=0}^{n-2} (n-1-\alpha_1, \alpha_1)!\ \overline{\mathcal{M}}_{0,\alpha_1+2} \times$$

$$\times \left(\coprod_{\beta_1+\beta_2=n-1-\alpha_1,\ \beta_i \geq 0,\ \beta_1>\beta_2} (n-1-\alpha_1-\beta_1, \beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1} \times \overline{\mathcal{M}}_{0,\beta_2+1} \coprod \right.$$

$$\left. \coprod_{2\beta_1=n-1-\alpha_1} \frac{1}{2}(\beta_1, \beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1} \times \overline{\mathcal{M}}_{0,\beta_1+1} \right) \coprod \overline{\mathcal{M}}_{0,n+1}$$

The two components of the inertia stack of $\overline{\mathcal{M}}_{1,n}$ when $n \geq 2$ endowed with the automorphisms ϵ and ϵ^5, are isomorphic to:

$$\overline{\mathcal{M}}_{0,n+1}$$

These components lie in the fiber of C_6 w.r.t the morphism neglecting sections.

Proof. The idea is the same as in the previous one, just much more complicated. The automorphisms are four instead of two. Let's look at the fixed points of:

$$x \longrightarrow \epsilon^{2n} x$$

$$y \longrightarrow \epsilon^{3n} y$$

on the curve C_6. when $n = 2, 4$ there are three fixed points: $(0, 1), (0, -1)$ and the infinity one. When $n = 1, 5$ there are no extra fixed points then infinity. So, the case with automorphisms ϵ and ϵ^5 is trivial: one just adds points over infinity. One crucial observation is that both the automorphisms ϵ and ϵ^5 exchange the two points $(0, 1)$ and $(0, -1)$, hence, for instance, the elliptic curve with a single marked point over $(0, 1)$ and the elliptic curve with a single marked point over $(0, -1)$ are the same inside the moduli space. Let's now show step by step how one finds out that formula.

- the case $n = 2$: $\overline{\mathcal{M}}_{0,3} \coprod \overline{\mathcal{M}}_{0,2} \times \overline{\mathcal{M}}_{0,2}$;
- the case $n = 3$: $\overline{\mathcal{M}}_{0,4} \coprod 2\overline{\mathcal{M}}_{0,3} \coprod \overline{\mathcal{M}}_{0,3}$;
- the case $n = 4$:

$$\overline{\mathcal{M}}_{0,5} \coprod 3\overline{\mathcal{M}}_{0,4} \coprod 3 \left(\overline{\mathcal{M}}_{0,3} \coprod \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \right) \coprod \left(\overline{\mathcal{M}}_{0,4} \coprod 3\overline{\mathcal{M}}_{0,3} \right).$$

- the case $n = 5$:

$$\overline{\mathcal{M}}_{0,6} \coprod 4\overline{\mathcal{M}}_{0,5} \coprod 6\overline{\mathcal{M}}_{0,4} \times \left(\overline{\mathcal{M}}_{0,3} \coprod \overline{\mathcal{M}}_{0,2}\right) \coprod$$

$$\coprod 4\overline{\mathcal{M}}_{0,3} \times \left(\overline{\mathcal{M}}_{0,4} \coprod 3\overline{\mathcal{M}}_{0,3}\right) \coprod$$

$$\coprod \left(\overline{\mathcal{M}}_{0,5} \coprod 3\overline{\mathcal{M}}_{0,4} \coprod 3\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}\right).$$

To perform the computations, one separates the marked points added to infinity from marked points added to the remaining two points. Adding marked points to $(0, i)$ and $(0, -i)$ one has to be aware that an elliptic curve with markings onto the first of this two is indistinguishible from the same elliptic curve with the same markings onto the second point. Hence if one has k marked points to put onto this two special points, the number of possibilities with k_1 onto the first is the binomial $(k - k_1, k_1)!$. The two k_1 and $k - k_1$ gives rise to the same object in the moduli space (one sees this using either the automorphism ϵ or ϵ^5), and so one has to count this two only once. When k is odd this is obtained simply imposing that $k_1 > k - k_1$. When k is even, one has to divide by two the case when $2k_1 = k$. □

These two lemmas allows the computation of the dimension of the cohomology for the twisted sector lying over the points of $\overline{\mathcal{M}}_{1,1}$ with non generic automorphism. The following lemma explains what happens adding extra marked points over the twisted sector $(\overline{\mathcal{M}}_{1,1}, -1)$.

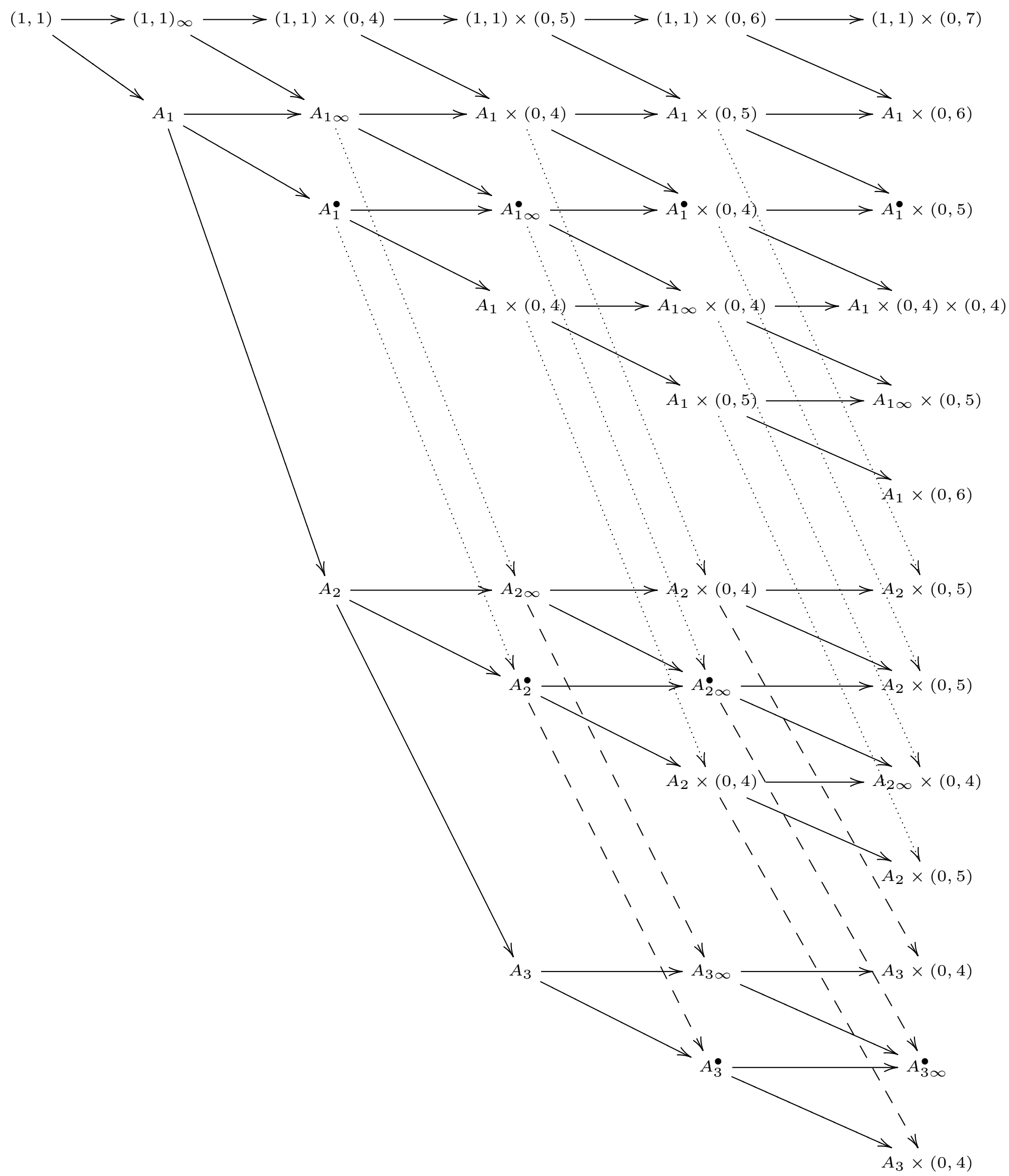
Lemma 1.22 (C)**.** *The last component of $I(\overline{\mathcal{M}}_{1,n})$, with the automorphism -1, obtained adding points over the component $(\overline{\mathcal{M}}_{1,1}, -1)$ of the inertia stack of the moduli space of elliptic curves with one marked point, is given by the following formula:*

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1, \alpha_2)!\ \overline{A_1} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$

$$\coprod_{\alpha_1+\alpha_2=n-3,\ \alpha_i\geq 0} ((\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2)!)\ \overline{A_2} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$

$$\coprod_{\alpha_1+\alpha_2=n-4,\ \alpha_i\geq 0} (2(\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2)!)\ \overline{A_3} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2} \coprod \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1}$$

Proof. The proof is as usual given with a tree. We will use the following notation: $\overline{\mathcal{M}}_{g,n}$ will be denoted (g, n). A moduli space with an extra ∞ point marked is indicated with a subscript ∞, while when a point different from ∞ is marked, it will be denoted by an upper dot.



As usual, each vertex has to be counted with its proper multiplicity, i.e. the number of possible paths to reach it from the starting vertex on the left. Arrows are of different kind just to help the reader. □

Let's now give some example, because the matter is a little cumbersome. To get knowledge of the marked points we refer to the trees inside the proofs of lemmas A,B and C.

Example 1.23. (The inertia stack of $\overline{\mathcal{M}}_{1,2}$)

$$I(\overline{\mathcal{M}}_{1,2}) = (\overline{\mathcal{M}}_{1,2}, 1) \coprod (\overline{\mathcal{M}}_{1,1}, -1) \coprod (\overline{A_1}, -1) \coprod 2\,(C_4, i/-i) \coprod (C_6, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5) \coprod$$
$$\coprod (C_6, \epsilon^2/\epsilon^4)$$

Example 1.24. (The inertia stack of $\overline{\mathcal{M}}_{1,3}$)

$$I(\overline{\mathcal{M}}_{1,3}) = (\overline{\mathcal{M}}_{1,3}, 1) \coprod (\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,4}, -1) \coprod 3\,(\overline{A_1}, -1) \coprod (\overline{A_2}, -1) \coprod$$
$$\coprod 2(C_4, i/-i) \coprod (C_4 \times \overline{\mathcal{M}}_{0,4}, i/-i) \coprod (C_6 \times \overline{\mathcal{M}}_{0,4}, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5) \coprod 4(C_6, \epsilon^2/\epsilon^4)$$
$$\coprod 3\,(C_4, i/-i) \coprod (C_4 \times \overline{\mathcal{M}}_{0,4}, i/-i)$$
$$\coprod (C_6 \times \overline{\mathcal{M}}_{0,4}, \epsilon/\epsilon^2/\epsilon^4/\epsilon^5) \coprod (C_6, \epsilon^2/\epsilon^4) \coprod (C_6, \epsilon^2/\epsilon^4) \coprod (C_6, \epsilon^2/\epsilon^4)$$

Theorem 1.25. *(Structure of the inertia stack: explicit description of the twisted sectors) The twisted sectors of the inertia stack of $\overline{\mathcal{M}}_{1,n}$ when $n > 3$ are, up to isomorphism and with the notations introduced in this chapter:*

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1, \alpha_2)!\ \overline{A_1} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$

$$\coprod_{\alpha_1+\alpha_2=n-3,\ \alpha_i\geq 0} ((\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2)!)\ \overline{A_2} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2}$$

$$\coprod_{\alpha_1+\alpha_2=n-4,\ \alpha_i\geq 0} (2(\alpha_1, \alpha_2)! + (\alpha_1+1, \alpha_2)!)\ \overline{A_3} \times \overline{\mathcal{M}}_{0,\alpha_1+2} \times \overline{\mathcal{M}}_{0,\alpha_2+2} \coprod \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,n+1} \coprod$$

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1,\alpha_2)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\times\overline{\mathcal{M}}_{0,\alpha_2+2}\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\coprod_{\alpha_1+\alpha_2=n-2,\ \alpha_i\geq 0} (\alpha_1+1,\alpha_2)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\times\overline{\mathcal{M}}_{0,\alpha_2+2}\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\coprod_{\alpha_1=0}^{n-2}(n-1-\alpha_1,\alpha_1)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\ \times$$

$$\times\left(\coprod_{\beta_1+\beta_2=n-1-\alpha_1,\ \beta_i\geq 0,\ \beta_1>\beta_2}(n-1-\alpha_1-\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_2+1}\coprod\right.$$

$$\left.\coprod_{2\beta_1=n-1-\alpha_1}\frac{1}{2}(\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_1+1}\right)\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\coprod_{\alpha_1=0}^{n-2}(n-1-\alpha_1,\alpha_1)!\ \overline{\mathcal{M}}_{0,\alpha_1+2}\ \times$$

$$\times\left(\coprod_{\beta_1+\beta_2=n-1-\alpha_1,\ \beta_i\geq 0,\ \beta_1>\beta_2}(n-1-\alpha_1-\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_2+1}\coprod\right.$$

$$\left.\coprod_{2\beta_1=n-1-\alpha_1}\frac{1}{2}(\beta_1,\beta_1)!\ \overline{\mathcal{M}}_{0,\beta_1+1}\times\overline{\mathcal{M}}_{0,\beta_1+1}\right)\coprod\overline{\mathcal{M}}_{0,n+1}\coprod$$

$$\overline{\mathcal{M}}_{0,n+1}\coprod\overline{\mathcal{M}}_{0,n+1}$$

For a more detailed description of the twisted sectors, and to know the automorphisms acting on each sector, we refer to lemmas [A], [B] and [C].

Theorem 1.26 ([BT])**.** *(Kunneth Formula) The singular cohomology with rational coefficients satisfies the following formula. If X and Y are topological spaces,*

$$H^n(X\times Y)\cong\bigoplus_{l+k=n}H^k(X)\otimes H^l(Y)$$

Theorem 1.27. *(Cohomology of $\overline{\mathcal{M}}_{0,n}$) [See [K]]. The canonical map from the Chow groups to homology (in characteristic zero):*

$$A_*(\overline{\mathcal{M}}_{0,n})\xrightarrow{cl}H_*(\overline{\mathcal{M}}_{0,n})$$

is an isomorphism. Moreover, there is a recursive formula for the Betti numbers of $\overline{\mathcal{M}}_{0,n}$. Finally, since we are interested in cohomology and all is taken to be with rational coefficients, the following recursive formula holds, where $a^k(n) := dim(H^{2k}(\overline{\mathcal{M}}_{0,n}))$:

$$a^k(n+1) = a^k(n) + a^{k-1}(n) + \frac{1}{2}\sum_{j=2}^{n-2}\binom{n}{k}\sum_{l=0}^{l=k-1} a^l(j+1)a^{k-1-l}(n-j-1)$$

$$a^k(3) = \begin{cases} 1 & \textit{if } k=0 \\ 0 & \textit{otherwise} \end{cases}$$

Remark 1.28. (cohomology of $\overline{\mathcal{M}}_{1,1}$). According to [AV] and to [B], one can compute the cohomology of $\overline{\mathcal{M}}_{1,1}$ also with integer coefficients. It turns out to be $\mathbb{Z}[t]/(24t^2)$ (this takes into account also the product structure). Tensoring with $\mathbb{Q}$, one finds out the usual graded vector space structure of the cohomology of $\mathbb{P}^1$, which is $\mathbb{Q}$ in degree 0 and 2, and zero in all other degrees.

Putting all together, we get our final result for the cohomology of $\overline{\mathcal{M}}_{1,n}$. Since the additive structure of $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ is completely known [Co3], this result giving the dimension for the twisted sectors, determines the dimension of the whole orbifold cohomology too.

2 Age

Let's compute first the age for $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$. Observe first that the nontwisted sector has no shift. Moreover, since the formula:

$$a(g,x) + a(g^{-1},x) = \text{codimension of } X_{(g)}$$

holds, the age of the twisted sector corresponding to the automorphism (-1) is $\frac{1}{2}$ the codimension. The components with the automorphism -1 are finite coverings of $\overline{\mathcal{M}}_{1,1}$, they have dimension 1. Hence the age of the previous components is easily computed as one half the codimension of the sector involved.

Moreover, the age for weighted projective spaces is known ([M], [AV]).

$$a(C_4, i) = 1 - a(C_4, -i) = \frac{1}{2}$$

$$a(C_6, \epsilon) = 1 - a(C_6, \epsilon^5) = \frac{1}{3} = a(C_6, \epsilon^4) = 1 - a(C_6, \epsilon^2)$$

This completes the computation of the shift for the smooth and stable case with one marked point.

Proposition 2.1. *(Age in the smooth case) In the following, we describe the age for the remaining components of the inertia stack of the moduli space of smooth elliptic curves* $\mathcal{M}_{1,n}$*:*

- *when* $n = 2$*:*
$$a(C_4, i) = \frac{5}{4} = 2 - a(C_4, -i)$$
$$a(C_6, \epsilon^2) = \frac{4}{3} = 2 - a(C_6, \epsilon^4);$$
- *when* $n = 3$*:*
$$a(C_6, \epsilon^2) = 2 = 3 - a(C_6, \epsilon^4).$$

Proof. We use the universal curve. The functor neglecting the last point is a smooth morphism between the algebraic stacks $\mathcal{M}_{1,2}$ and $\mathcal{M}_{1,1}$. Take the two étale atlases X and Y respectively for the two spaces $\mathcal{M}_{1,2}$ and $\mathcal{M}_{1,1}$. The morphism now is a smooth morphism between smooth algebraic varieties. Then the following sequence of vector spaces is exact:

$$0 \longrightarrow T_p f^{-1}(f(p)) \longrightarrow T_p(X) \longrightarrow T_{f(p)}Y \longrightarrow 0$$

The action of the two elements of the group $i, -i \in \mu_4$ onto $T_p(X)$ splits up into an action over the two sides. Therefore the age is the sum of the two ages. i acts over C_4:

$$x \longrightarrow -x$$
$$y \longrightarrow -iy$$

Then tangent point at the fixed locus of the curve $(0,0)$ is given by the equation $x = 0$, and the action of i on it is given by $-i$. Conversely, one checks easily that the action of $-i$ on it is given by i. As regards the points C_6, the fixed points are $(0,1)$ and $(0,-1)$ as we have already stressed in the previous section. The tangent space embedded in $\mathbb{A}^2$ are given by the parametrizations $(t,1)$ and $(t,-1)$ respectively. The actions of ϵ^2, ϵ^4 are given by the multiplication by ϵ^4 and ϵ^2 respectively.

□

Thanks to the proposition and to all previous considerations, age is completely known in the smooth case. In the stable case, we do not know age for twisted sectors over the isolated points C_4 and C_6.

Theorem 2.2. *(Local-Global exact sequence of Ext) The following is exact:*

$$0 \longrightarrow H^1(X, \mathfrak{Hom}(\mathcal{F}, \mathcal{G})) \longrightarrow Ext^1(\mathcal{F}, \mathcal{G}) \longrightarrow H^0(\mathfrak{Ext}^1(\mathcal{F}, \mathcal{G})) \longrightarrow$$

$$\longrightarrow H^2(X, \mathfrak{Hom}(\mathcal{F}, \mathcal{G})) \longrightarrow Ext^2(\mathcal{F}, \mathcal{G})$$

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