ERRATUM TO: THE ORBIFOLD COHOMOLOGY OF MODULI OF HYPERELLIPTIC CURVES

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The statement of [2, Proposition 3.3] contains an error: some of the twisted sectors of $[\mathcal{M}_{0,n}/S_n]$ appear quotiented by an extra, erroneous, action of S_2, the symmetric group on two elements. This error propagates then in the main result [2, Theorem 4.3], and in [2, Theorem 5.1] and [2, Corollary 5.3]. From now on, we will omit the reference to [2].

The error occurs during the proof of Proposition 3.3, when considering the cases $a = 0, 2$, and $N > 2$. When the automorphism α is not an involution, its action on the (co)tangent spaces to the fibers distinguishes between the two branch points in C'; conversely, to reconstruct the automorphism α one needs the ordering of the two branch points in C'. The corrected description of the twisted sectors of $[\mathcal{M}_{0,n}/S_n], n \geq 3$, is therefore as follows.

(1) Suppose $N > 2$, or n odd. If there exists $a \in \{0, 1, 2\}$ such that $n = kN + a$:

$$I_N([\mathcal{M}_{0,n}/S_n]) = \begin{cases} \coprod_{\chi \in \widetilde{\mu}_N^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a = 0, 2 \\ \coprod_{\chi \in \mu_N^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a = 1 \end{cases}$$

and $I_N([\mathcal{M}_{0,n}/S_n])$ is empty otherwise.

(2) If n is even, $n =: 2g + 2$:

$$I_2([\mathcal{M}_{0,n}/S_n]) = (\mathcal{M}_{0,g+2}/S_g \times S_2, -1) \coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, -1).$$

(The stacky description of the rigidified inertia stack is obtained by substituting $\mathcal{M}_{0,k+2}/S$ with the *stack* quotients $[\mathcal{M}_{0,k+2}/S]$, as already observed in Remark 3.4.)

We propose here a non-trivial check of the correctness of the delicate, albeit relatively elementary, result of this corrected version of Proposition 3.3. For a stack X we have two notions of Euler characteristics: the (usual) topological Euler characteristic $\chi(X)$ of the associated coarse moduli space, and the *orbifold* Euler characteristic $e(X)$. The latter is not necessarily an integer, and when G is a finite group acting on a scheme X, such that $X = [Y/G]$, it satisfies $|G| \cdot e(X) = \chi(Y)$. It is well-known that these two quantities are related by the equality

$$\chi(X) = e(I(X)), \tag{1}$$

see [1, p.21]. The latter equality provides a consistency check of the corrected version of Proposition 3.3, together with Remark 3.4. Let us fix $X = [\mathcal{M}_{0,n}/S_n]$: after multiplying by

$\frac{1}{N}$ the orbifold Euler characteristic of each rigidified I_N, the right hand side of (1) becomes

$$\frac{-1}{n(n-1)(n-2)} + \frac{(-1)^{\frac{n}{2}}}{2(n-2)} - \frac{(-1)^{\frac{n}{2}}}{2n} - \sum_{\substack{N \text{ divides } (n-a),\\ a\in\{0,1,2\}, N>2}} (-1)^{\frac{n-a}{N}} \frac{\phi(N)}{\frac{(-1)^a+3}{2}(n-a)}, \text{ for } n \text{ even;}$$

$$\frac{1}{n(n-1)(n-2)} - \sum_{\substack{N \text{ divides } (n-a),\\ a\in\{0,1,2\}, N>1}} (-1)^{\frac{n-a}{N}} \frac{\phi(N)}{\frac{(-1)^a+3}{2}(n-a)}, \text{ for } n \text{ odd.}$$

A direct computation shows that both these expressions equal 1: the left hand side of (1). Here we used that

$$\chi(\mathcal{M}_{0,n}/S_n) = 1, \quad e(\mathcal{M}_{0,n}) = (-1)^{n+1}(n-3)!;$$

and the following elementary formula for the Euler totient function ϕ:

$$\sum_{d \text{ divides } n} (-1)^{\frac{n}{d}} \phi(d) = \begin{cases} 0 & n \text{ even}, \\ -n & n \text{ odd}. \end{cases}$$

Let us now restate the corrected version of Theorem 4.3 in view of the error we observed above. For all $g \geq 2$, we describe the twisted sectors of $\mathcal{H}_g$, the moduli stack of smooth hyperelliptic curves of genus g. Besides the untwisted sector, $I_1^{red}(\mathcal{H}_g)$ contains one more copy of $\mathcal{M}_{0,2g+2}/S_{2g+2}$, corresponding to the hyperelliptic involution. Let us now describe the cases when $N \geq 2$.

(1) Suppose $N > 2$. If there exists $a \in \{0,1,2\}$ such that $2g+2 = kN + a$:

$$I_N^{red}(\mathcal{H}_g) = \begin{cases} \coprod_{\chi\in\widetilde{\mu}_N^*, \lambda\in\pm 1} (\mathcal{M}_{0,k+2}/S_k, (\chi,\lambda)) & a=0, \ k \text{ even} \\ \coprod_{\chi\in\mu_N^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a=0, \ k \text{ odd} \\ \coprod_{\chi\in\mu_N^*\sqcup\mu_{2N}^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a=1 \\ \coprod_{\chi\in\widetilde{\mu}_{2N}^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a=2, \ k \text{ even}, \ N \text{ even} \\ \coprod_{\chi\in\widetilde{\mu}_N^*\sqcup\widetilde{\mu}_{2N}^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a=2, \ k \text{ even}, \ N \text{ odd} \\ \coprod_{\chi\in\overline{\mu}_{2N}^*} (\mathcal{M}_{0,k+2}/S_k, \chi) & a=2, \ k \text{ odd} \end{cases}$$

and $I_N^{red}(\mathcal{H}_g)$ is empty otherwise.

(2) if g is even:

$$I_2^{red}(\mathcal{H}_g) = (\mathcal{M}_{0,g+2}/S_g, -1) \coprod (\mathcal{M}_{0,g+3}/S_{g+1}, -1).$$

(3) if g is odd:

$$I_2^{red}(\mathcal{H}_g) = (\mathcal{M}_{0,g+2}/S_g \times S_2, \zeta_4) \coprod (\mathcal{M}_{0,g+2}/S_g \times S_2, \zeta_4^3) \coprod$$
$$\coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, (-1,1)) \coprod (\mathcal{M}_{0,g+3}/S_{g+1} \times S_2, (-1,-1)).$$

Another small mistake occurs in Remark 4.3: it is not correct that the above is a stack-theoretic description of the rigidified inertia stack of $\mathcal{H}_g$, even after substituting each occurrence of $\mathcal{M}_{0,k+2}/S$ with the *stack* quotient $[\mathcal{M}_{0,k+2}/S]$. Indeed, in the presentation above, we have further rigidified the twisted sectors along the hyperelliptic involution.

We observe that the consistency check we used above does not produce any further check on the corrected version of Theorem 4.3. Indeed, the most delicate point of the latter theorem is understanding when the double covers $I'(f)_N$ are trivial and when they are not, a distinction that does not affect the orbifold Euler characteristic.

In view of the original error in Proposition 3.3 observed above, and of its consequences on Theorem 4.3, the orbifold Poincaré polynomial of moduli of smooth hyperelliptic curves of Theorem 5.1 is given by the formula

$$P^{CR}_{\mathcal{H}_g}(q) = \sum_{(k,N,i)\in A_{2g+2}} q^{a_g(i,N)} P^0_{k+2;k,1,1}(q) + \sum_{(k,N,i)\in A_{2g+1}} 2q^{b_g(i,N)} P^0_{k+2;k,1,1}(q) +$$

$$+ \sum_{(k,N,i)\in A_{2g}} q^{b_g(i,N)} P^0_{k+2;k,1,1}(q) + 2 + \begin{cases} q^{\frac{g-1}{2}} P^0_{g+3;g+1,1,1}(q) + q^{\frac{g}{2}} P^0_{g+2;g,1,1}(q) & \text{if } g \text{ is even} \\ 2q^{\frac{g-1}{2}} P^0_{g+3;g+1,2}(q) + 2q^{\frac{g}{2}} P^0_{g+2;g,2}(q) & \text{if } g \text{ is odd,} \end{cases}$$

where the sets of indices are as in Theorem 5.1. The formula for the dimension of the orbifold cohomology of $\mathcal{H}_g$, contained in Corollary 5.5, must be corrected as follows.

(1) If g is even, $n = 2g+2$:

$$h_{CR}(g) = 3 + 2g + 2 \sum_{\substack{n=kN+a,\\ N>2, a\in\{0,1,2\}}} k\phi(N).$$

(2) If g is odd, $n = 2g+2$:

$$h_{CR}(g) = 2 + 4\left(\lfloor\frac{n-2}{4}\rfloor + \lfloor\frac{n-1}{4}\rfloor\right) + 2 \sum_{\substack{n=kN+a,\\ N>2, a\in\{0,1,2\}}} k\phi(N).$$

In the cases of $g = 2, 3$, the corrected version of all proposition differs from the original one only in the stacky world. This follows after observing that, for $\mathcal{M}_{0,k+2}/S_k$ with $k = 1, 2$, quotienting by the action of S_2 on the last two points does not change the coarse space.

The correction of the original error and of its consequences (and of few more typos) are subsumed in the revised version of my paper [2] posted on arXiv.

References

[1] Kai Behrend, *Cohomology of stacks*, Intersection theory and moduli, 249–294 (electronic), ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[2] Nicola Pagani, *The orbifold cohomology of moduli of hyperelliptic curves*, Int. Math. Res. Not. IMRN 2012, no. 10, 2163–2178, arXiv:1011.4946.

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