Sums of small fractional parts and a problem of Littlewood

We prove estimates for certain partial sums of which the simplest nontrivial examples are

$$\sum_{n=1}^{N} \frac{1}{\|\alpha n\|\|\beta n\|}, \quad \text{and} \quad \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{1}{\|\alpha m + \beta n\|}.$$

Here α and β are real numbers, and $\|x\|$ is the distance from the real number x to the nearest integer. Such estimates are somewhat related to a notorious open problem of Littlewood: is it true that for all pairs of real numbers α and β we have

$$\liminf_{n \to \infty} n\|\alpha n\|\|\beta n\| = 0?$$

We also consider estimates for more general sums that contain products of many linear forms in many variables. If time permits we will also discuss an analogue of such problems in function fields. This is joint work with Thái Hoàng Lê.