**Two-Degrees-of-Freedom Systems**

One-degree-of-freedom systems allow basic concepts, such as frequency, damping, initial conditions, resonance and phase, etc., to be introduced and appreciated. However, real systems have infinite number of degrees-of-freedom. On many occasions, these systems can be approximated as having finite degrees-of-freedom, the simplest of which has two degrees-of-freedom.

Multi-degrees-of-freedom systems afford a venue for introducing eigenvectors(*modes*), matrices and vibration absorption.

**Free Vibration**

A mass-spring system:

*k*3

*k*2

*k*1

*m*2

*m*1

Free-body diagrams

*k*2(*x*1-*x*2)

*k*2(*x*1-*x*2)

*k*3*x*2

*k*1*x*1

*x*1

*m*1

*m*2

*x*2

Equations of motion

  or  (1)

In matrix form as

  (2)

or

 

Following the procedure for one-degree-of-freedom systems, assume the solution in the form of

 

This leads to

  (3)

The condition for a non-trivial solution of  to exist is

  (4)

or

  (5)

Notice ** in equation (5) is the natural frequency (frequencies)

This is known in linear algebra as an (generalized) eigenvalue problem. For matrices of very low orders (2×2, 3×3, 4×4), ‘hand’ calculation is feasible. For higher-orders matrices, suitable algorithms and computer programs must be used.

For the above simple eigenvalue problem, the determinant method gives (characteristic equation)

 

or

 

This quadratic equation has two roots of . Suppose  and . The above characteristic equation become



The two analytic solutions are  and .

Equation (3) allows the ratio of the two amplitudes to be found as

 

Substitution of one eigenvalue a time into equation (3) yields

 For :  For : 

Mode shapes

0.73

1

-2.73

1

*a*1  *a*2

*a*1

*a*2

**Proportionality**

A mode shape is represented by a vector, called *eigenvector* mathematically. The absolute magnitude of individual elements of an eigenvector is indefinite ⎯ an eigenvector multiplied by a *scalar* is still an eigenvector. But the **relative proportion** of the individual elements of an eigenvector is fixed. For this reason, an eigenvector is normally normalized (scaled according to a rule) and then becomes unique. Normalisation helps to graphically display a mode, compare modes and facilitate computation.

Orthogonality

$x\_{i}^{T}Mx\_{k}^{}=\left\{\begin{array}{c}\ne 0 i=k\\=0 i\ne k\end{array}\right.$$x\_{i}^{T}Kx\_{k}^{}=\left\{\begin{array}{c}\ne 0 i=k\\=0 i\ne k\end{array}\right.$

**Recall orthogonality between two vectors.**

**Normalisaton**

1. Let the maximum element be one and the other elements scaled as 
2. Multiply a factor to an eigenvector so that  (mass-normalisation)
3. Scale an eigenvector so that .

A normalised eigenvector is called a ***normal eigenvector***.

The two-degree-of-freedom system may vibrate in any one of the two modes. In general, the motion of the system is a ***linear combination*** of all modes as



Factors *A*1 and *A*2, phase angles , like in one-degree-of-freedom systems, should be determined by the *initial conditions*.

**Initial Conditions**

For the linear system, given initial conditions as

 

Substituting them into



leads to

 

The four unknowns can be determined and the solution is

 



For a system of *n* degrees-of-freedom, **M** and **K** are *n*×*n* square matrices. There usually exist *n* natural frequencies  and *n* eigenvectors , by solving equation (5). The motion can be written as



**Rotational Systems**

*k*1 *k*2 *k*3

** 1 **2

*J*1 *J*2

 *k*1**1 *k*2(**1*-*2) *k*3**2

** 1 **2

*J*1 *J*2

 

**Coupled Pendulum**



*m*

*m*

*a*

*l*

Two natural frequencies and modes can be found as

 

**Damped Vibration**

*k*1

*k*3

*k*2

*m*2

*m*1

*c*



It is no longer valid to assume that  due to presence of damping. In general, the solution can be assumed as , where ** is complex. This leads to

 

Suppose  and . The above equation becomes

 

Introduce a new variable . A new equation appears as

 

Two roots at : ; eigenvalues:  .