Surgery in Complex Dynamics.

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Introduction

Magic Lecture 1:
Cut and Past surgery using interpolations.

Magic Lecture 2:
Quasi-conformal massage, -maps on drugs -.

Magic Lecture 3:
Quasi-conformal surgery using real conjugacies.

Magic Lecture 4:
Trans-quasi-conformal surgery.

Announcement

There are OPEN CODY Ph.d positions at RUC in Denmark

Recapitulation on q-c mappings.

Let U be an open subset of $\mathbb{C}$ and let $W^{1,p}_{\mathrm{loc}}$, $1 \leq p$ denote the Sobelev space of $L^1_{\mathrm{loc}}(U)$ functions with distributional derivatives in $L^p_{\mathrm{loc}}(U)$, i.e. the $L^1_{\mathrm{loc}}(U)$ function $\phi : U \longrightarrow \mathbb{C}$ belongs to $W^{1,p}_{\mathrm{loc}}$ if there are functions $f, g \in L^p_{\mathrm{loc}}(U)$ such that for all test (smooth) functions h with compact support in U:

$$\int_U h \cdot f dz d\overline{z} = -\int_U h_z \cdot \phi dz d\overline{z},$$

$$\int_U h \cdot g dz d\overline{z} = -\int_U h_{\overline{z}} \cdot \phi dz d\overline{z}.$$

In this case we write $\phi_z = f$ and $\phi_{\overline{z}} = g$.

Recapitulation on q-c mappings II

An o-p homeomorphism $\phi : U \longrightarrow V$, $U, V \subset \mathbb{C}$ belonging to $W^{1,1}_{\text{loc}}(U)$ is differentiable almost everywhere and its Jacobian $\text{Jac}(\phi) = |\phi_z|^2 - |\phi_{\overline{z}}|^2$ belongs to $L^1_{\text{loc}}(U)$.

A Beltrami differential on $U \subset \mathbb{C}$ is a $(-1,1)$-form $\mu = \mu(z)\frac{d\overline{z}}{dz}$, where $\mu : U \longrightarrow \mathbb{C}$ belongs to $L^{\infty}_{\text{loc}}(U)$and $|\mu(z)| < 1$ a.e.
A Beltrami differential is naturally an ellipse field and is also called an almost complex structure previously denoted σ.

We say that μ is integrable if there exists a homeomorphism $\phi : U \longrightarrow V$, $U, V \subset \mathbb{C}$ belonging to $W^{1,1}_{\text{loc}}(U)$ solving the Beltrami equation:

$$\phi_{\overline{z}} = \mu \cdot \phi_z \qquad \text{a.e.}$$

Recapitulation on q-c mappings II

If $||\mu||_\infty = k < 1$, then ϕ is called quasi conformal and:

a) ϕ belongs to $W^{1,p}_{\text{loc}}(U)$ for some $p > 2$ and in particular for $p = 2$,

b) ϕ is unique up to post composition by a holomorphic map,

c) ϕ is absolutely continuous, i.e. for any measureable set $K \subset U$: $\mathrm{area(K)} = 0$ implies $\mathrm{area}(\phi(\mathrm{K})) = 0$,

d) ϕ 's inverse ϕ^{-1} is quasi conformal with the same k.

Trans-q-c mappings

The bound $||\mu_1||_\infty = k$ is equivalent to the hyperbolic bound

$$\mathrm{d}_{\mathbb{D}}(0, \mu(z)) = \log \frac{1+|\mu(z)|}{1-|\mu(z)|} \leq \log \frac{1+k}{1-k} = \log K,$$

where $K = \dfrac{1+k}{1-k}$. Define $K_\mu(z) = \dfrac{1+|\mu(z)|}{1-|\mu(z)|}$.
It may seem as a harsh condition to ask that K_μ should be bounded in order for μ to be integrable.
And in fact it turns out that there is room for improvements, but not much.
Guy David proved that there is a class of (hyperbolically) unbounded Beltrami differentials, which are uniquely integrable up to post composition by a holomorphic map.

t-q-c mappings II

Let us say that a Beltrami differential $\mu : U \longrightarrow \mathbb{C}$ is a David-Beltrami differential if there exists constants $M > 0, \alpha > 0$ and $K_0 > 1$ such that

$$\forall K > K_0 : \mathrm{area}(\{\mathrm{z} \in \mathrm{U} | \mathrm{K}_\mu(\mathrm{z}) = \frac{1 + |\mu(\mathrm{z})|}{1 - |\mu(\mathrm{z})|} > \mathrm{K}\}) \leq \mathrm{M}\mathrm{e}^{-\alpha \mathrm{K}}.$$

or equivalently (when U has finite area) there exists $\alpha' > 0$ such that

$$\int_U \mathrm{e}^{\alpha' K_\mu(z)} dz d\overline{z} < \infty.$$

David proved the following integration theorem for David-Beltrami differentials:

t-q-c mappings III

Theorem 1. *(David) Any David-Beltrami differential* $\mu : U \longrightarrow \mathbb{C}$*, where* $U \subset \mathbb{C}$ *is a domain is integrable by some o-p homeomorphism* $\phi : U \longrightarrow V$ *in* $W^{1,1}_{\mathrm{loc}}$*,* $V \subset \mathbb{C}$*. Such a map is called a David map or a trans quasi conformal map. And:*

a) ϕ *belongs to* $W^{1,p}_{\mathrm{loc}}(U)$ *for every* $p < 2$*,*

b) ϕ *is unique up to post composition by a holomorphic map, i.e. if* $\Phi : U \longrightarrow V'$ *in* $W^{1,1}_{\mathrm{loc}}$ *is another solution to the same Beltrami equation then there exists a holomorphic function* $f : V \longrightarrow V'$ *with* $\Phi = f \circ \phi$*.*

c) ϕ *is absolutely continuous,*

d) ϕ^{-1} *is in general not a David map.*

t-q-c mappings IV

The notion of David maps naturally generalize to maps of $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$.
Note howerver that on a Riemann surface we must insist that a Beltrami-form is a $(-1,1)$ form $\mu(z)\frac{d\overline{z}}{dz}$ and the the Beltrami equation is an equation on forms:

$$\phi_{\overline{z}}\, d\overline{z} = \mu\, \frac{d\overline{z}}{dz}\, \phi_z\, dz$$

However as long as the support of μ is a compact subset of $\mathbb{C}$, with the canonical chart z, we need not worry about forms.

These were the tools from analysis, here are applications:

Applications

Theorem 2. *(P. & Zakeri) There exists a full measure subset $\mathcal{E} \subset [0,1] \setminus \mathbb{Q}$ such that for all $\theta \in \mathcal{E}$ the quadratic polynomial $P_\theta(z) = \mathrm{e}^{i2\pi\theta}z + z^2$ has a Siegel disk with Jordan boundary containing the critical point and with locally connected Julia set of zero area.*

The main “new” ingredients relative to the q-c surgery on f_θ leading to F_θ in the previous lecture is that: We construct for $\theta \in \mathcal{E}$ a David (instead of a q-c) extension $H_\theta : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ of the conjugacy h_θ. We prove that the F_θ^* invariant Beltrami form μ_θ is a David-Beltrami form. And we prove that the integrating David homeomorphism $\phi_\theta : (\overline{\mathbb{C}}, \mu_\theta) \longrightarrow (\overline{\mathbb{C}}, 0)$ conjugates F_θ to P_θ.

Applications

Theorem 3. *(Haissinsky) There exists a David homeomorphism $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ and a domain $U \subset \Lambda_0(1/2)$ with $Q_0(U) \subset U$ such that*

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus U & \xrightarrow{Q_0} & \overline{\mathbb{C}} \\ \phi \downarrow & & \downarrow \phi \\ \overline{\mathbb{C}} \setminus \phi(U) & \xrightarrow{Q_{1/4}} & \overline{\mathbb{C}}. \end{array}$$

where $Q_c(z) = z^2 + c$ and where $\phi_{\overline{z}} = 0$ on $\overline{\mathbb{C}} \setminus \Lambda_0(1/2)$.

Arithmetic Conditions

Definition 4. *Recall that we may write any irrational number* $\theta \in [0,1]$ *in a unique way as a continued fraction with positive integer coefficients (called partial fractions):*

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}.$$

And recall that the number θ is said to be of bounded type if the sequence (a_n) is bounded.

Arithmetic Conditions II

The set of bounded type numbers has zero Lebesgue measure in $[0,1]$. In fact:

Theorem 5. *(Borel, Bernstein, Khinchin) Let* $\psi : \mathbb{N} \longrightarrow]0,\infty]$ *be any function.*

a) if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\psi(n)} < \infty$ *then for almost every* $0 < \theta < 1$ *there are only finitely many* n *for which* $a_n(\theta) > \psi(n)$*.*

b) if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\psi(n)} = \infty$ *then for almost every* $0 < \theta < 1$ *there are infinitely many* n *for which* $a_n(\theta) > \psi(n)$*.*

Arithmetic Conditions III

In the following I shall outline the proof of Theorem 2 above.

We prove that Theorem 2 holds for $\mathcal{E}$ the set of irrational numbers $\theta \in [0, 1]$ with

$$\log(a_n(\theta)) = \mathcal{O}(\sqrt{n}).$$

By the previous theorem this set is certainly of full measure. However it is a subset of the Diophantine numbers of exponent d for any $d > 2$.

Cubic Blaschke product Revisited

Recall the the previous lecture the cubic Blaschke product

$$f(z) = z^2 \frac{z+3}{1+3z}$$

which restricts to a real analytic circle homeomorphism with a critical fixed point at 1 and which further has super attracting fixed points at 0 and ∞ of local degree 2.
Let $\theta \in [0,1] \setminus Q$ be given and let $\eta = \eta(\theta) \in]0,1[$ be such that the map

$$f_\theta(z) := \mathrm{e}^{i2\pi\eta} \cdot f(z)$$

has rotation number θ on the unit circle.

Blaschke Revisited II

Let $h = h_\theta : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the unique conjugacy of f_θ to R_θ with $h(1) = 1$.

Theorem 6. *(P. & Zakeri, Yoccoz) For every $\theta \in \mathcal{E}$ there exists a David extension $H : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ of h, with $H(0) = 0$ i.e. a homeomorphism in $W^{1,1}_{\text{loc}}(\mathbb{D})$ solving the Beltrami equation for some David-Beltrami differential μ on $\mathbb{D}$.*

As before we define a new dynamical system $F_\theta = F_{\theta,H} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$

$$F_\theta(z) = \begin{cases} f_\theta(z), & z \notin \mathbb{D}, \\ H^{-1} \circ R_\theta \circ H(z) & z \in \mathbb{D} \end{cases}$$

A David-Beltrami form in $\mathbb{D}$

Let $\mu = H^*(0)$, then $F^*(\mu)(z) = \mu(z)$ for $z \in \mathbb{D}$. We extend μ to an F^* invariant Beltrami differential μ on $\overline{\mathbb{C}}$ by defining

$$\mu(z) = \begin{cases} (f^n)^*(\mu) = \mu((f^n)(z)) \cdot \frac{\overline{(f^n)'(z)}}{(f^n)'(z)}, & z \in F^{-n}(\mathbb{D}) \\ 0, & z \notin \cup_{n \geq 0} F^{-n}(D) \end{cases}$$

where $f = f_\theta$. In the quasi conformal case we where then ready to apply Ahlfors-Bers because $K_\mu(z) = K_\mu(f(z))$ for all $z \notin \mathbb{D}$, so that boundedness in $\mathbb{D}$ implies the same bound everywhere. However in order for μ to be a David-Beltrami differential we need that the areas of large dilatation decreases exponentially with the dilatation.

Area Distortion bounds

However f distorts areas unboundedly, so potentially we could have small areas blow up too much under backwards iteration!!

Let G detnote the set of all possible branches g of f^{-n} on $\mathbb{D}$ such that $f^k \circ g(\mathbb{D}) \cap \mathbb{D} = \emptyset$ for $0 \leq k < n$ and define a measure ν on $\mathbb{D}$ by:

$$\forall E \subset \mathbb{D},\ \text{measureable}: \quad \nu(E) = \mathrm{area(E)} + \sum_{\mathrm{g \in G}} \mathrm{area(g(E))}.$$

Then

Area Distortion bounds II

Theorem 7. *(P,Zakeri) The measure ν is dominated by a power of the Lebesgue measure on $\mathbb{D}$. That is there exists $\beta, 0 < \beta < 1$ independent of θ and a constant $C = C(\theta)$ such that for every measureable set $E \subset \mathbb{D}$:*

$$\nu(E) \leq C(\mathrm{area(E)})^{\beta}.$$

With this theorem at hand we see that μ is indeed a David-Beltrami form compactly supported on $\mathbb{C}$ whenever its restriction to $\mathbb{D}$ is a David-Beltrami form. Since we have

$$\forall K > K_0 : \mathrm{area}(\{\mathrm{z} \in \mathbb{C} | \mathrm{K}_{\mu}(\mathrm{z}) = \frac{1 + |\mu(\mathrm{z})|}{1 - |\mu(\mathrm{z})|} > \mathrm{K}\}) \leq \mathrm{CM}^{\nu} \mathrm{e}^{-\alpha \nu \mathrm{K}}$$

and $\mu = 0$ on a neighbourhood of ∞.

Integrating μ

We may hence integrate μ and proceed as usual. We let $\phi : (\overline{\mathbb{C}}, \mu) \longrightarrow (\overline{\mathbb{C}}, 0)$ be an integrating homeomorphism for μ normalized by $\phi(\infty) = \infty$, $\phi(0) = 0$ and $\phi(1) = -\lambda_\theta/2$.

We define $P(z) = \phi \circ F \circ \phi^{-1}$ and we want to check that P is holomorphic.
We need to check that $\phi \circ F \in W^{1,1}_{\text{loc}}(\overline{\mathbb{C}} \setminus \{1\})$ so that both ϕ and $\phi \circ F$ solve the Beltrami equation for μ and thus $\phi \circ F = P \circ \phi$ with P holomorphic.

$\phi \circ F \in W^{1,1}_{\text{loc}}(\overline{\mathbb{C}} \setminus \{1\})$

On $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$: $F = f$ is holomorphic so that $\phi \circ F \in W^{1,1}_{\text{loc}}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$. On $\mathbb{D}$ we have $F = H^{-1} \circ R_\theta \circ H$ and thus

$$\phi \circ F = \phi \circ H^{-1} \circ R_\theta \circ H.$$

But both ϕ and H integrate μ on $\mathbb{D}$ so that $\phi \circ H^{-1}$ is holomorphic. Hence $\phi \circ F$ on $\mathbb{D}$ equals the post composition of H with a conformal map. Hence also $\phi \circ F \in W^{1,1}_{\text{loc}}(\mathbb{D})$. Finally we just need to check that if ψ is a homeomorphism on some domain U, L is a line through U and $\psi \in W^{1,1}_{\text{loc}}(U \setminus L)$ then infact $\psi \in W^{1,1}_{\text{loc}}(U)$.This is a fun little exercise.

Proof completion

To complete the proof define $J_F = \partial\Lambda_0(\infty)$.
Note that $\phi(J_F) = J_P$.
The remainder of the theorem follows from

Theorem 8. *(P) For every* $\theta \in [0,1] \setminus \mathbb{Q}$ *the set* J_F *is locally connected and of zero area.*

Let me conclude we a few problem suggestions:

Similar applications?

Is it true that:

Question 1. *For almost all* θ*, any cubic polynomial* $P_{\theta,a}(z) = \lambda_\theta z + az^2 + z^3$ *has a Siegel disk* $\Delta_{\theta,a}$ *whose boundary contains at least one and in particular cases both the finite critical points of* $P_{\theta,a}$*?*

Question 2. *For almost all* $0 < \theta, \tau < 1$ *with* $\theta + \tau \neq 1$ *the quadratic rational map*

$$R_{\theta,\tau}(z) = z\frac{z + \lambda_\theta}{1 + \lambda_\tau z}$$

has Jordan Siegel disks with disjoint closure $\Delta^0_{\theta,\tau}$ *and* $\Delta^\infty_{\theta,\tau}$ *around zero and* ∞ *respectively, and with each Jordan boundary containing a critical point?*

Questions II

Question 3. *Is there a wider condition than a) and b) below for which the same statement, holds, except with t-q-c instead of q-c?*
Suppose the Blaschke product B restricts to an o-p degree $d \geq 2$ covering $B : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and that
a) no critical point $c \in \mathbb{S}^1$ for B is recurrent to a critical point on $\mathbb{S}^1$,
b) every periodic point for B on $\mathbb{S}^1$ is repelling.
Then there exists a rational map R and an o-p q-c homeomorphism $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that

$$\phi \circ B = R \circ \phi : \overline{\mathbb{C}} \setminus \mathbb{D} \longrightarrow \overline{\mathbb{C}},$$

and $\Lambda = \phi(\mathbb{D})$ is a super attracting basin for R on which R is conformally conjugate to z^d.

Questions III

Is it true that

Question 4. *For almost all irrational* θ *the parabolic quadratic polynomial* $Q_{\frac{1}{4}}(z) = z^2 + \frac{1}{4}$ *has a virtual-Siegel disk* Δ *with rotation number* θ *and Jordan boundary containing the critical point?*

Is it true that

Question 5. *For almost all irrational* θ *the map* $\lambda_\theta \sin z$ *has a Siegel disk* Δ_θ *with Jordan boundary containing the two nearest critical points* $\pm\pi/2$*, but none of the others?*