Surgery in Complex Dynamics.

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Introduction

Magic Lecture 1:

Cut and Past surgery using interpolations.

Magic Lecture 2:

Quasi-conformal massage, -maps on drugs -.

Magic Lecture 3:

Quasi-conformal surgery using real conjugacies.

Magic Lecture 4:

Trans-quasi-conformal surgery.

Cubic Blaschke products

For $3 \leq r$ the cubic Blaschke product

$$f(z) = z^2 \frac{z+r}{1+rz}$$

restricts to a real analytic circle homeomorphism (diffeo for $3 < r$) with a fixed point at 1 of multiplier $(3-r)/(1-r)$. It further has super attracting fixed points at 0 and ∞ of local degree 2.
Fix $r \geq 3$ and let $\theta \in [0,1] \setminus \mathbb{Q}$ be given. Then there exists a (unique) $\eta = \eta(\theta) \in]0,1[$ such that the map

$$f_\theta(z) := \mathrm{e}^{i2\pi\eta} \cdot f(z)$$

has rotation number θ on the unit circle.

Rotation number

Definition and Theorem 1 (Poincaré). *Suppose $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is an o-p homeomorphism and let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of $g(\mathrm{e}^{i2\pi z})$ to $\mathrm{e}^{i2\pi z}$. Then for any $x \in \mathbb{R}$ the limit*

$$t = \lim_{n\to\infty} \frac{G^n(x) - x}{n}$$

exists and does not depend on x. Moreover the fractional part $\theta = [t] \in \mathbb{R}/\mathbb{Z}$ is independent of the choice of lift G. It is called the rotation number of g.
Furthermore let $R_\theta(z) = \mathrm{e}^{i2\pi\theta} z$, and let $w_0 \in \mathbb{S}^1$ be arbitrary. Then there exists a unique o-p semi-conjugacy $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $h(w_0) = 1$ of g to R_θ:

$$h \circ g = R_\theta \circ h.$$

Actual conjugacy

Theorem 2. *Let $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be an o-p homeomorphism with irrational rotation number. Iif g is either:*
a) (Denjoy) a C^2 diffeomorphism or
b) (Yoccoz) a real analytic homeomorphism with at least one critical point. Then the Poincaré semi-conjugacy is a conjugacy.

Note that if $r = 3$ then condition b) is fulfilled and

if $r > 3$ then condition a) is fulfilled.

Hence in either case f_θ is topologically topologically conjugate to R_θ by a unique o-p homeomorphism $h = h_\theta$, which is say normalized by fixing $h(1) = 1$.

Blaschke surgery

Choose a homeomorphic extension $H = H_\theta : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ of the circle conjugacy h say with $H(0) = 0$.

Define a new dynamical system $F_\theta = F_{\theta,H} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$

$$F_\theta(z) = \begin{cases} f_\theta(z), & z \notin \mathbb{D}, \\ H^{-1} \circ R_\theta \circ H(z) & z \in \mathbb{D} \end{cases}$$

Then F_θ is a topological degree 2 map which is conformal outside $\overline{\mathbb{D}}$, but it is certainly not conformal on $\overline{\mathbb{D}}$.
However if H is quasi conformal, then F is quasi regular and we are in buisness:

Blaschke Surgery 2

Suppose for the time being that H is q-c. Let σ denote the unique F^* invariant almost complex structure on $\overline{\mathbb{C}}$, satisfying

$$\sigma(z) = \begin{cases} H^*(\sigma_0), & z \in \mathbb{D} \\ \sigma_0, & z \notin \cup_{n \geq 0} F^{-n}(D). \end{cases}$$

Let $\phi : (\overline{\mathbb{C}}, \sigma) \longrightarrow (\overline{\mathbb{C}}, \sigma_0)$ be the integrating o-p q-c homeomorphism which is normalized by $\phi(0) = 0$ and $r\mathrm{e}^{i2\pi\eta}\phi(z)/z \to 1$ as $z \to \infty$.
This construction was invented in the case $r > 3$ by Etienne Ghys in his thesis and ported to the critical case $r = 3$ by Douady in his 1987 Seminaire Bourbaki.

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Then $P_\theta(z) = \phi \circ F_\theta \circ \phi^{-1}$ is a quadratic polynomial with fixed point at 0 of multiplier λ_θ and with $P_\theta(z)/z^2 \to 1$ as $z \to \infty$. That is $P_\theta(z) = \lambda_\theta z + z^2$.

Moreover P_θ has a Siegel disk $\Delta_\theta = \phi(\mathbb{D})$ with quasi-circle boundary $\phi(\mathbb{S}^1)$.

Finally if $r = 3$ so that 1 is a critical point, then the quasi-circle boundary of Δ_θ contains the critical point $-\lambda_\theta/2$ and if $r > 3$ then it does not.

Quasi-conformal extension?

Recall that

Definition 3. *A homeomorphism $F : \Gamma \longrightarrow \gamma$, where $\Gamma, \gamma \subset \mathbb{C}$ is called quasi-symmetric if there exists a constant $C > 1$ such that for any triplet of points $x, y, z \in \Gamma$ with $|x - y| = |y - z|$:*

$$1/C \leq \frac{|F(x) - F(y)|}{|F(y) - F(z)|} \leq C.$$

A classical result due to Ahlfors and Beurling is that

Theorem 4. *A homeomorphism $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has a quasi conformal extension $H : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ if and only if h is quasi-symmetric.*

So the exiting question is if h_θ is q-s?

Conditions for quasi-symmetri

Michel Herman (who was supervisor for Etienne Ghys) proved first that there exists $r > 3$ and irrational rotation numbers θ such that f_θ is q-s conjugate to R_θ, but not analytically conjugate.

Ghys used this and his construction to prove that there are quadratic polynomials with a Siegel disk, whose boundary is a quasi-circle not containing the critical point.

This was/is interesting, because Douady previously had proved that. If P_θ has a Siegel disk Δ_θ and if the critical point $-\lambda_\theta/2$ does not belong to $\overline{\Delta_\theta}$. Then the Julia set J_θ is not locally connected.

Herman-Swiatek

Herman who attended the Bourbaki seminar of Douady proved within a few days after the seminar the following theorem, using ideas of Grazyk Swiatek:

Theorem 5 (Herman-Swiatek). *Let* $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ *be areal analytic, o-p circle homeomorphism with a critical point. Then the Poincaré-Yoccoz conjugacy is quasi-symmetric if and only if* θ *is of bounded type.*

Thus we may conclude:

Theorem 6. *For* θ *of bounded type the quadratic polynomial* $P_\theta(z) = \lambda_\theta z + z^2$ *has a Siegel disk, whose boundary is a quasi-circle containing the critical point.*

Remark the converse is also true, but the proof is more elaborate.

Bounded type

Definition 7. *We may write any irrational number $\theta \in [0,1]$ in a unique way as a continued fraction with positive integer coefficients (called partial fractions):*

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \frac{1}{\ddots}}}}$$

The number θ is said to be of bounded type if the sequence (a_n) is bounded.

The prime example is the golden mean with

$$a_n \equiv 1.$$

Similar applications

Theorem 8. *(S. Zakeri) For θ of bounded type any cubic polynomial $P_{\theta,a}(z) = \lambda_\theta z + az^2 + z^3$ has a Siegel disk $\Delta_{\theta,a}$ whose boundary contains at least one and in particular cases both the finite critical points of $P_{\theta,a}$.*

Theorem 9. *(M. Yampolsky and S. Zakeri) For $0 < \theta, \tau < 1$ of bounded type with $\theta + \tau \neq 1$ the quadratic rational map*

$$R_{\theta,\tau}(z) = z\frac{z + \lambda_\theta}{1 + \lambda_\tau z}$$

has a Siegeld disks $\Delta^0_{\theta,\tau}$ and $\Delta^\infty_{\theta,\tau}$ around 0 and ∞ respectively, with disjoint quasi circle boundaries each of which contains a critical point.

Similar applications II

Theorem 10. *(L. Ma) Suppose the Blaschke product B restricts to an o-p degree $d \geq 2$ covering $B : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and that*
a) every critical point $c \in \mathbb{S}^1$ for B is not recurrent to a critical point,
b) every periodic point for B on $\mathbb{S}^1$ is repelling.
Then there exists a rational map R and an o-p q-c homeomorphism $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus \mathbb{D} & \xrightarrow{B} & \overline{\mathbb{C}} \\ \phi \big\downarrow & & \big\downarrow \phi \\ \overline{\mathbb{C}} \setminus \phi(\mathbb{D}) & \xrightarrow{R} & \overline{\mathbb{C}}. \end{array}$$

and $\Lambda = \phi(\mathbb{D})$ is a super attracting basin for R on which R is conformally conjugate to z^d.

More refined similar applications

Theorem 11. *(A. Cheritat) For any* θ *of bounded type the parabolic quadratic polynomial* $Q_{\frac{1}{4}}(z) = z^2 + \frac{1}{4}$ *has a virtual-Siegel disk* Δ *with rotation number* θ *and quasi-circle boundary containing the critical point.*

Theorem 12. *(G. Zhang) For any* θ *of bounded type the map* $\lambda_\theta \sin z$ *has a Siegel disk* Δ_θ *with quasi-circle boundary containing the two nearest critical points* $\pm\pi/2$*, but none of the others.*