Surgery in Complex Dynamics.

Carsten Lunde Petersen

IMFUFA at NSM, Roskilde University

Introduction

Magic Lecture 1:

Cut and Past surgery using interpolations.

Magic Lecture 2:

Quasi-conformal massage, -maps on drugs -.

Magic Lecture 3:

Quasi-conformal surgery using real conjugacies.

Magic Lecture 4:

Trans-quasi-conformal surgery.

Wringing and stretching

Wringing and stretching the complex structure

Let $s \in \mathbb{H}_+$ and define $\widetilde{l}_s : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\widetilde{l}_s(z) = \frac{1}{2}(s+1)z + \frac{1}{2}(s-1)\overline{z} = (s-1)z_x + z = sz_x + iz_y,$$

where $z = z_x + iz_y$. Or written as an $\mathbb{R}$-linear selfmap of $\mathbb{R}^2$:

$$\widetilde{l}_s \begin{pmatrix} z_x \\ z_y \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ s_y & 1 \end{pmatrix} \begin{pmatrix} z_x \\ z_y \end{pmatrix}.$$

Evidently $\widetilde{l}_s$ is an o-p q-c homeo with constant dilatation

$$\widetilde{\mu}_s(z) = \widetilde{\mu}_s = \frac{s-1}{s+1}.$$

Wringing and Stretching II

Being an $\mathbb{R}$-linear map $\widetilde{l}_s(z) = sz_x + iz_y$ commutes with real multiplication and conjugates translation by $\Lambda \in \mathbb{C}$ to translation by $\widetilde{l}_s(\Lambda)$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto rz} & \mathbb{C} \\ \widetilde{l}_s \downarrow & & \downarrow \widetilde{l}_s \\ \mathbb{C} & \xrightarrow{z \mapsto rz} & \mathbb{C}. \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto z+\Lambda} & \mathbb{C} \\ \widetilde{l}_s \downarrow & & \downarrow \widetilde{l}_s \\ \mathbb{C} & \xrightarrow{z \mapsto z+\widetilde{l}_s(\Lambda)} & \mathbb{C}. \end{array}$$

Furthermore $\widetilde{l}_s(iy) = iy,$ and $\widetilde{l}_s(x) = sx$ and thus

$$\widetilde{l}_s(i\mathbb{R}) = i\mathbb{R}, \qquad \widetilde{l}_s(\mathbb{R}) = s\mathbb{R} \quad \text{and} \quad \widetilde{l}_s(\mathbb{R} + iy) = s\mathbb{R} + iy.$$

Wringing and Stretching III

For $s \in \mathbb{H}_+$ consider likewise the map $l_s : \mathbb{C} \longrightarrow \mathbb{C}$ given by

$$l_s(z) = z \cdot |z|^{s-1} = \frac{z}{|z|}|z|^s$$

An easy computation shows that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widetilde{l}_s} & \mathbb{C} \\ e^z \big\downarrow & & \big\downarrow e^z \\ \mathbb{C}^* & \xrightarrow{l_s} & \mathbb{C}^*. \end{array}$$

Wringing and Stretching IV

It immediately follows that l_s:

a) is an o-p q-c homeo with a coherent dilatation given by

$$\mu_s(z) = \frac{z}{\overline{z}} \frac{s-1}{s+1} = \frac{\overline{\log'(z)}}{\log' z} \cdot \widetilde{\mu}_s(\log(z)).$$

b) fixes pointwise the unit circle and maps the radial line $\exp(\mathbb{R} + iy)$ to the logaritmic spiral $\exp(s\mathbb{R} + iy)$,

c) commutes with $z \mapsto z^d$ for $d \in \mathbb{N}$ and conjugates multiplication by $\lambda \in \mathbb{C}^*$ to multiplication by $l_s(\lambda)$.

d) for any z the map $s \mapsto \mu_s(z)$ is holomorphic (Möbius).

Wringing and Stretching V

We define a Lie-group structure $\star$ on $\mathbb{H}_+$ by requirering that the map $s \mapsto l_s$ is a contravariant group homomorphism, that is

$$l_{s \star s'} = l_{s'} \circ l_s.$$

An easy computation shows that this amounts to

$$\begin{aligned} s \star s' &= s'_x s_x + i(s'_y s_x + s_y) \\ &= s_x s' + i s_y \\ &= l_{s'}(s) \end{aligned}$$

When stressing this group structure we shall write $\mathcal{R}$ for $\mathbb{H}_+$.

Wringing and Stretching VI

The Lie-group $(\mathcal{R}, \star)$ is non-commutative, but it is generated by the two one-dimensional commutative subgroups:

Stretch: $(\mathbb{S}, \star) = (\mathbb{R}_+, \star)$ and

Wring: $(\mathcal{W}, \star) = (1 + i\mathbb{R}, \star)$.

These two subgroups act on $\mathcal{R}$ from left by multiplication and by addition of the imaginary part repsectively:

$$\begin{aligned} s = s_x \in \mathbb{S}, &\qquad s' \in \mathcal{R} \Rightarrow s \star s' = ss', \\ w = 1 + iw_y \in \mathcal{W}, &\qquad s' \in \mathcal{R} \Rightarrow w \star s' = s' + iw_y. \end{aligned}$$

Druging a map with an attracting orbit

Theorem 1 (Douady and Hubbard). *For any hyperbolic component* H *of the Mandelbrotset* M *the multiplier map* $\lambda : H \longrightarrow \mathbb{D}$ *for the associated attracting periodic orbit is an isomorphism*

Theorem 2 (Eremenko and Lyubich). *For any hyperbolic component* H *of the exponential family* $E_\kappa(z) = \exp(z + \kappa)$ *the multiplier map* $\lambda : H \longrightarrow \mathbb{D}^*$ *for the associated attracting periodic orbit is a universal covering map.*

Druging II

Common features:

In both cases we can define a group action by $(\mathcal{R}, \star)$ on the subset of H consisting of parameters for which the multiplier map is non-zero.

In the quadratic case let H^* denote H deprived of its center(s). Applying this action to a fixed parameter in H^* defines a surjective and periodic map onto H^*. Passing to a quotient space essentially yields the result.

In the exponential case applying this action to any parameter yields an injective and surjective map and hence the result.

The action

Initially you may choose to think of f_1 as either a quadratic polynomial Q_c or an exponetial map f_κ.

Suppose $f_1^k(z_1) = z_1$ and $(f_1^k)'(z_1) = \lambda_1 \in \mathbb{D}^*$.
Let Λ_1 denote the immediate basin of z_1. We shall assume the singular (0)/ critical (c_1) value belongs to Λ_1.
Let $\phi_1 : \Lambda_1 \longrightarrow \mathbb{C}$ denote the linearizer say normalized by $\phi_1(0) = \lambda_1$ (exponential) / $\phi_1(c_1) = \lambda_1$ (quadratic).
For $s \in \mathcal{R}$ let σ_s denote the unique f_1^* invariant almost complex structure, which satisfies:

$$\sigma_s = (\mathrm{l}_s \circ \phi_1)^*(\sigma_0) \qquad \text{on} \qquad \Lambda_1,$$

$$\sigma_s = \sigma_0 \qquad \text{on} \qquad \overline{\mathbb{C}} \setminus \cup_{n \geq 0} f_1^{-n}(\Lambda_1)$$

The action II

Let $h_s : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be an integrating o-p q-c homeomorphism for σ_s.

As normalization conditions we take $h_s(\infty) = \infty$ and $h_s(0) = 0$ in both cases, however:

If $f_1(z) = \exp(z + \kappa_1)$ we demand $h_s(i2\pi) = i2\pi$.

If $f_1(z) = z^2 + c_1$ we demand $h_s(z)/z \to 1$ as $z \to \infty$. In the later case we find easily that

$$f_s(z) := h_s \circ f_1 \circ h_s^{-1} = z^2 + c_s, \qquad \text{where} \qquad c_s = h_s(c_1).$$

The action III

In the first exponential case we need first to realize that σ_s satisfies $\sigma_s(z + i2\pi) = \sigma_s(z)$, in order to conclude that $h_s(z + i2\pi) = h_s(z) + i2\pi$. Then it follows easily that

$$f_s(z) := h_s \circ f_1 \circ h_s^{-1} = h_s(\mathrm{e}^{\kappa_1}) \cdot \mathrm{e}^z.$$

Taking $\kappa_s = \log(h_s(\mathrm{e}^{\kappa_1}))$ to be the unique continuous and thus holomorphic choice of logarithm sending 1 to κ_1, we have

$$f_s(z) = \exp(z + \kappa_s).$$

The action IV

We have holomorphic mappings into parameter spaces $s \mapsto \kappa_s, c_s : \mathbb{H}_+ \to \mathbb{C}$ such that

$$f_s^k(z_s) = z_s := h_s(z_1), \qquad \text{and} \qquad \left(f_s^k\right)'(z_s) = \lambda_s = l_s(\lambda).$$

Where either

$$f_s(z) = \exp(z + \kappa_s) \qquad \text{or} \qquad f_s(z) = z^2 + c_s.$$

These mappings into parameterspaces are in fact group actions on H^*:

$$\forall s, s' \in \mathcal{R} : \kappa_{s \star s'} = \left(\kappa_s\right)_{s'}, \qquad \text{and} \qquad c_{s \star s'} = \left(c_s\right)_{s'}.$$

Exploiting the group action

Notice first that by the group action if some map $g_1 = f_s$ for some $s \in \mathcal{R}$, then $g_{s^{-1}} = f_1$ and the two group action image sets are equal:

$$\{f_s | s \in \mathcal{R}\} = \{g_s | s \in \mathcal{R}\}.$$

Moreover this set is a neighbourhood of f_1 (and of g_1). Since H^* is connected it follows that

$$H^* = \{f_s | s \in \mathcal{R}\}$$

Secondly by the group action and the above it suffices to consider the case $\lambda_1 = \mathrm{e}^{-1}$ so that

$$\lambda_s = l_s(\lambda_1) = \mathrm{e}^{-s}.$$

Exploiting the group action II

Thirdly by the group action property if $f_{s'} = f_s$ for some $s \neq s' \in \mathcal{R}$ then $f_{s' \star s^{-1}} = f_1$ and hence $s_1 = s' \star s^{-1} = 1 + n2\pi i$ for some $n \in \mathbb{Z}^*$ as $\mathrm{e}^{-1} = \lambda_1 = \lambda_{s_0} = \mathrm{e}^{-s_0}$.

Moreover $f_s = f_{s_1 \star s}$ for all $s \in \mathcal{R}$. Suppose $s_1 = 1 + n2\pi i$ satisfies $f_s = f_{s_1 \star s}$ for all $s \in \mathcal{R}$. Then $s_k = s_1^{k\star} = 1 + kn2\pi i$ also satisfies $f_s = f_{s_k \star s}$ for all $s \in \mathcal{R}$.

Thus we can suppose $s_1 = 1 + n2\pi i$ with $n \in \mathbb{N}$ minimal.

Exploiting the group action II

Then the surjective, holomorphic and $n2\pi i$ periodic map $s \mapsto f_s$ from $\mathbb{H}_+$ to H^* descends to a biholomorphic map $F = w \mapsto \widehat{f}_w : Dstar \to H^*$ given by $\widehat{f}_w = f_{-n \log w}$ and the multiplier map λ on H^* is the n-fold covering $(F^{-1})^n$ of $\mathbb{D}^*$.

In the case of quadratic polynomials one can prove that the above holds with $n = 1$ so that the multiplier map is an isomorphism and extends to a biholomorphic map of H onto $\mathbb{D}$.

In the case of the exponential family, which we shall pursue here we need to prove that there is no n such that $f_{s_1} = f_1$ with $s_1 = 1 + n2\pi i$:

Proof of Eremenko-Lyubich Th.

Suppose to the contrary that for some $n \in \mathbb{N}$

$$f_{s_1} = f_1 \qquad \text{with} \qquad s_1 = 1 + n2\pi i.$$

Then by the above there is a biholomorphic map $K : \mathbb{D}^* \longrightarrow H$ such that $E_{K(w)} = \exp(z + K(w)) = f_{-n\log w}$ has a k-periodic orbit of mulitplier w^n.

That is H is a punctured disk with puncture $\kappa_0 \in \overline{\mathbb{C}}$. Let us first notice that for any $\kappa > 0$, the singular value 0 of E_κ iterates to ∞ so that $\mathbb{R}_+ \cap H = \emptyset$. Thus $\kappa_0 \neq \infty$.

Proof of Eremenko-Lyubich Th. II

Let $z_0(\kappa), \ldots z_{k-1}(\kappa)$ denote the attracting k-periodic orbit of E_κ, $\kappa \in H$. As κ approaches κ_0 the multiplier of this orbit converge to 0 by the above.

Hence some point say $z_0(\kappa)$ converges to 0 and $\Re(z_{k-1}(\kappa)) \to -\infty$. However by continuity

$$z_{k-1}(\kappa) = E_\kappa^{k-1}(z_0(\kappa)) \to E_{\kappa_0}^{k-1}(0) \in \mathbb{C}, \quad \text{as} \quad \kappa \to \kappa_0.$$

Thus $s \mapsto f_s$ is injective in the exponential case.

That is $s \mapsto \kappa(s) : \mathbb{H}_+ \to H$ is biholomorphic and the multiplier map $\lambda(\kappa) = \mathrm{e}^{-s(\kappa)}$ is a universal covering.