Surgery in Complex Dynamics.

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Introduction

Magic Lecture 1:

Cut and Past surgery using interpolations.

Magic Lecture 2:

Quasi-conformal massage, -maps on drugs -.

Magic Lecture 3:

Quasi-conformal surgery using real conjugacies.

Magic Lecture 4:

Trans-quasi-conformal surgery.

Terminology

Let $Q_c(z) = z^2 + c, \qquad c \in \mathbb{C}$,

and let $J_c := \partial\{z | Q_c^n(z) \longrightarrow \infty\} \subset \overline{\mathbb{C}}$ denote the Julia set.

Let $M := \{c \in \mathbb{C} | Q_c^n(0) \not\to \infty, \text{as} n \to \infty\}$ denote the Mandelbrot set.

A hyperbolic component H of M is a maximal domain on which Q_c has an attracting periodic orbit.

A center of a H is a parameter $c_0 \in H$ such that the corresponding periodic orbit has multiplier 0.

Surgery application 1

Mãne, Sad and Sullivan caracterized the hyperbolic component of a hyperbolic map f as the set of all maps which are q-c conjugate to f on the Julia set.

Theorem 1. *Any hyperbolic component H of M has a center.*

The following proof was initially done by Sullivan.

We shall give it in two steps:

Step 1: Period $k = 1$.

Step 2: Arbitrary period.

Cut and Glue in an attracting basin.

Suppose $Q_c(z_0) = z_0$ and $Q_c'(z_0) = \lambda \in \mathbb{D}$. Let Λ_0 denote the basin of z_0. Choose $U \subset \Lambda_0$ a topological disk with real analytic Jordan boundary, $c \in U$ and $Q_c(U) \subset\subset U$.

Then $z_0 \in U$ and $U' = Q_c^{-1}(U) \supset\supset U$ is a topological disk with real analytic Jordan boundary and $Q_c : \partial U' \longrightarrow \partial U$ is a degree 2 o-p covering. Let $\eta : U \longrightarrow \mathbb{D}(r)$ for some $0 < r < 1$ be a uniformization with $\eta(z_0) = 0$.

Define a new dynamics $F : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by

$$F(z) = \begin{cases} Q_c(z) & z \notin U', \\ \text{q-c interpolation} & z \in A' = U' \setminus \overline{U}, \\ \eta^{-1}((\eta(z))^2) & z \in \overline{U} \end{cases} .$$

A q-c gluing-lemma.

Definition 2. *An arc* $\gamma \subset \mathbb{C}$ *is called a quasi arc if there exists* $C > 0$ *such that for any pair of points* $x, y \in \gamma$ *the (smaller) subset* I *of* $\gamma \setminus \{x, y\}$ *bounded by* x *and* y *satisfies the 'no turning condition':*

$$\mathrm{diam}(I) \leq C|x - y|.$$

In particular any smooth compact arc is a quasi arc. A quasi-circle is a Jordan quasi arc.

Lemma 3. *Suppose* $f : U \longrightarrow V$ *is a homeomorphism between open subsets of* $\mathbb{C}$ *and that* $\gamma \subset \Omega$ *is a quasi arc. If* f *is q-c on* $\Omega \setminus \gamma$*, then* f *is q-c on* Ω*.*

F is quasi-regular

Definition 4. *A mapping $F : U \longrightarrow V$, $U, V \subset \overline{\mathbb{C}}$ is called quasi-regular, if locally it can be written as a composition $F = f \circ \phi$ where f is holomorphic and ϕ is q-c.*

Lemma 5. *The cut and glued mapping*

$$F(z) = \begin{cases} Q_c(z) & z \notin U', \\ \text{\textit{q-c interpolation}} & z \in A' = U' \setminus \overline{U}, \\ \eta^{-1}((\eta(z))^2) & z \in \overline{U} \end{cases}$$

is quasi-regular.

A new invariant almost complex structure

Let σ_0 denote the standard (Euclidean) almost complex structure and let σ denote the unique $F*$ invariant almost complex structure which coincides with σ_0 on U and $\overline{\mathbb{C}} \setminus \Lambda_0$.

Let $\phi : (\overline{\mathbb{C}}, \sigma) \longrightarrow (\overline{\mathbb{C}}, \sigma_0)$ denote the unique integrating o-p q-c homeomorphism for σ normalized by

$$\phi(z_0) = 0, \qquad \frac{\phi(z)}{z} \to 1 \qquad \text{as} \qquad z \to \infty.$$

Then $P = \phi \circ F \circ \phi^{-1} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is the quadratic polynomial $Q_0(z) = z^2$ and ϕ conjugates Q_c to $Q_0 = P$ on the complement of U' and in particular on a neighbourhood of J_c.

The higher case period

Lets pause and think about the more general higher period case.

A variant of Ahlfors-Beurling

Definition 6. *A homeomorphism $F:\Gamma\longrightarrow\gamma$, where $\Gamma,\gamma\subset\mathbb{C}$ is called quasi-symmetric if there exists a constant $C>1$ such that for any triplet of points $x,y,z\in\Gamma$ with $|x-y|=|y-z|$:*

$$1/C\leq\frac{|F(x)-F(y)|}{|F(y)-F(z)|}\leq C.$$

Remark that any q-c maping is quasi-symmetric.

Lemma 7. *Suppose $A',A''\subset\mathbb{C}$ are closed annuli with quasi-circle boundary components, oriented counter clockwise and $F:\partial A'\longrightarrow\partial A''$ is a locally quasi-symmetric $d:1$ covering respecting orientation and inner respectively outer boundary.*
Then F has a quasi-regular extension $F:A'\longrightarrow A''$ of degree d.

Polynomial-like mappings

Definition 8. *A polynomial like mapping is a proper holomorphic map $f : U' \longrightarrow U$ with prefered domain $U' \simeq \mathbb{D}$ and range $U \simeq \mathbb{D}$, where $U' \subset\subset U \subset \mathbb{C}$.*

We assume $\deg(f) = d \geq 2$ in order to make f dynamically interesting.

The filled-in Julia set K_f of f is the compact and non empty set

$$K_f = \{z \in U' | \forall n \in \mathbb{N} : f^n(z) \in U'\}.$$

Equivalences of polynomial-like mappings

Definition 9. *Two polynomial-like mappings* $f : U' \longrightarrow U$ *and* $g : V' \longrightarrow V$ *are q-c* *equivalent* *if*

a) there exists neighbourhoods Ω_f *and* Ω_g *of* K_f *and* K_g *respectively such that* $f : \Omega'_f \longrightarrow \Omega_f$ *and and* $g : \Omega'_g \longrightarrow \Omega_g$ *are polynomial-like where* $\Omega'_f = f^{-1}(\Omega_f)$, $\Omega'_g = g^{-1}(\Omega_g)$ *and*

b) there exists a q-c homeomorphism $\phi : \Omega_f \longrightarrow \Omega_g$ *such that*

$$\begin{array}{ccc} \Omega'_f & \xrightarrow{f} & \Omega_f \\ \phi \downarrow & & \downarrow \phi \\ \Omega'_g & \xrightarrow{g} & \Omega_g . \end{array}$$

Equivalences cont.

Definition 10. *The q-c equivalence* ϕ

$$\begin{array}{ccc} \Omega'_f & \xrightarrow{f} & \Omega_f \\ \phi\downarrow & & \downarrow\phi \\ \Omega'_g & \xrightarrow{g} & \Omega_g. \end{array}$$

is called

i) a hybrid equivalence if $\overline{\partial}\phi = 0$ *a.e. on* K_f.

ii) an external equivalence if $\overline{\partial}\phi = 0$ *a.e. on* $\Omega_f \setminus K_f$.

iii) a conformal equivalence if $\overline{\partial}\phi = 0$ *a.e. on* Ω_f *or equivalently* ϕ *is conformal.*

The Straightening Theorem

Surgery example 2

Note that a polynomial has polynomial-like restrictions of the same degree and shall henceforth also be considered a polynomial-like mapping.

Theorem 11. *Let* $f : U' \longrightarrow U$ *be polynomial-like of degree* $d \geq 2$*. Then there exists a polynomial* P *of degree* d *hybridly equivalent to* $f : U' \longrightarrow U$*.*
Moreover if K_f *is connected, then* P *is unique up to affine conjugacy.*

Shishikuras rotation ring to disk surgery

Surgery example 3

Theorem 12 (Shishikura). *Suppose the rational map* $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *has a Herman-ring* H *with core geodesic* γ *and let* D_1 *and* D_2 *denote the two connected components of* $\overline{\mathbb{C}} \setminus \gamma$*. Then there exist rational maps* R_i*,* $i =, 1, 2$ *each with a fixed Siegel disk* Δ_i *and o-p q-c homeomorphisms* $\phi_i : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *conformal on* $D_i \cap H$ *and conjugating* R *to* R_i *on* D_i*.*

Augmentation of the Julia set

Theorem 13 (Branner and Douady). *There is a natural embedding of the* $1/2$*-limb of the Mandelbrot set, mapping the* $1/2$*-vein,* $[-2, \frac{1}{4}]$ *onto the* $1/4$*-vein in particular the later is an arc.*