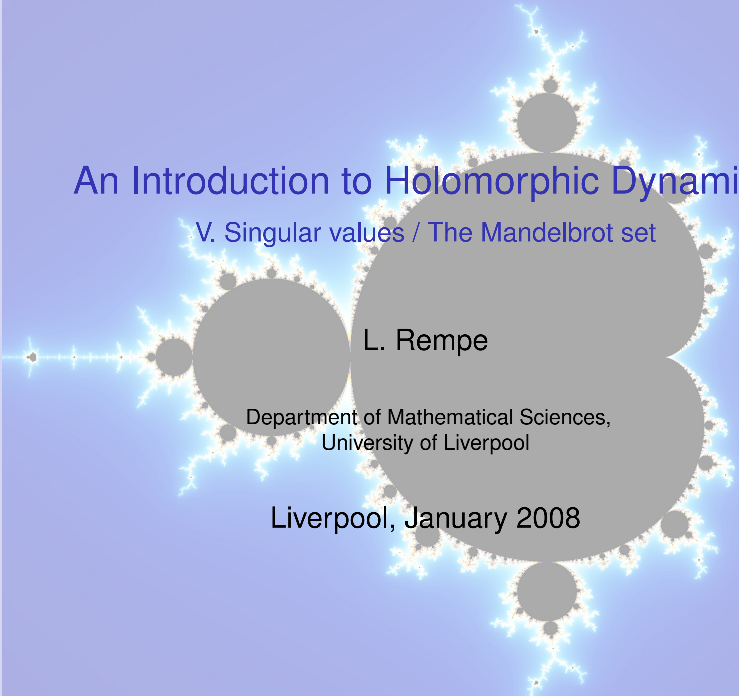

An Introduction to Holomorphic Dynamics

V. Singular values / The Mandelbrot set

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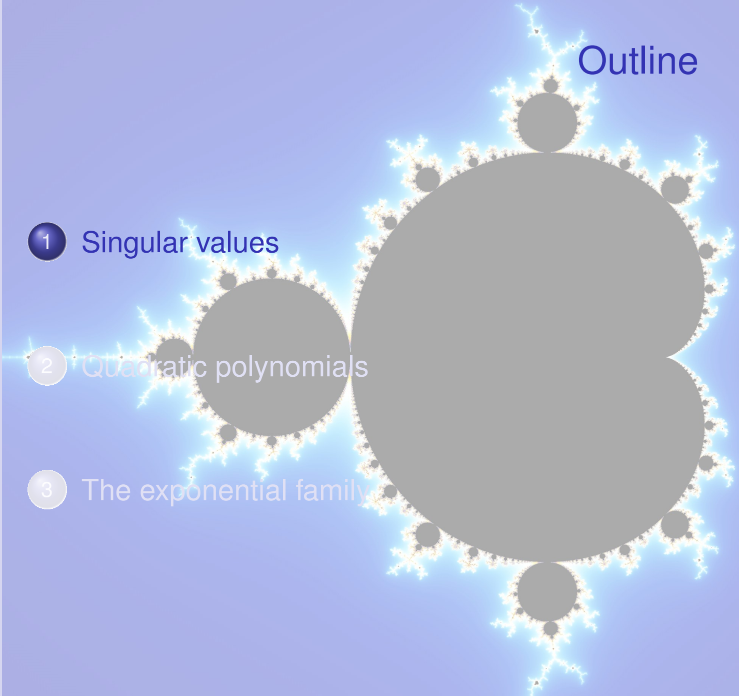
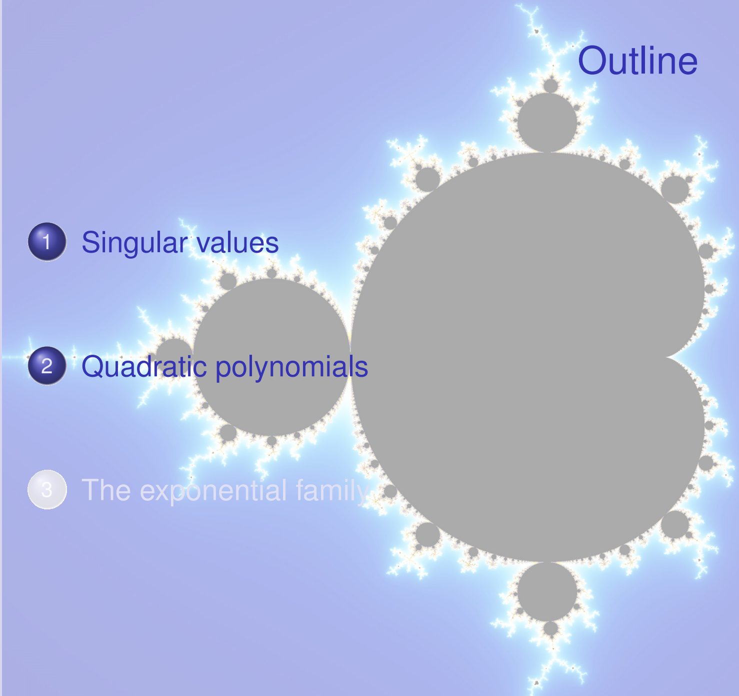
Liverpool, January 2008

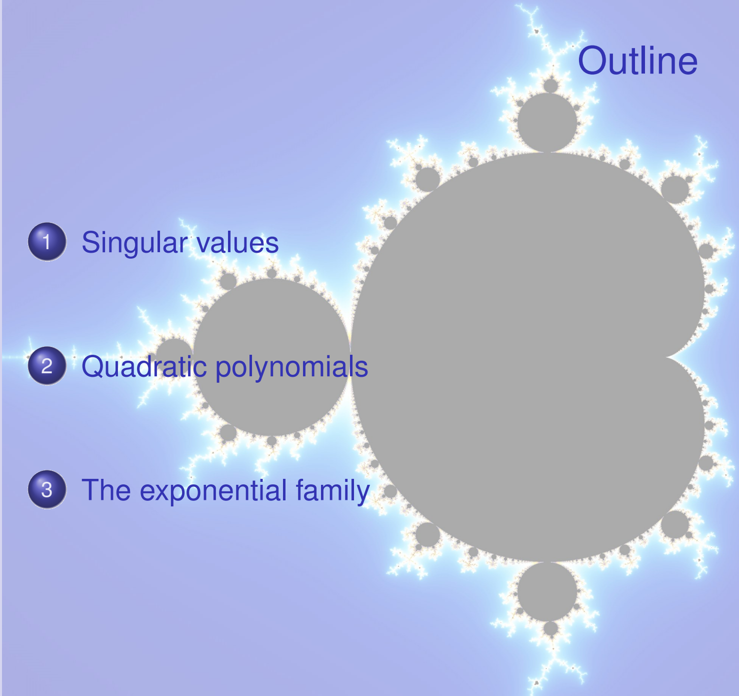
Outline



1. Singular values
2. Quadratic polynomials
3. The exponential family

Outline





Outline

Singular values

The set $\text{sing}(f^{-1})$ contains all values in which some branch of f^{-1} cannot be defined. There are two types of such points:

- v is a critical value if $v = f(c)$, $f'(c) = 0$.
- a is an asymptotic value if

  there is a curve $\gamma : (0, 1] \to \mathbb{C}$ such that $\lim_{t\to 0} |\gamma(t)| = \infty$ and $\lim_{t\to 0} f(\gamma(t)) = a$.

- $S(f) := \overline{\text{sing}(f^{-1})}$ is the set of singular values

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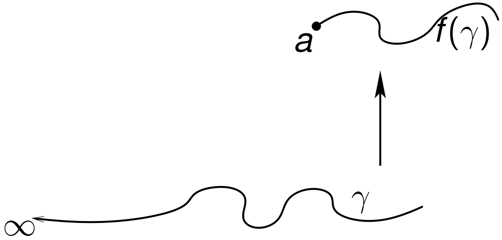

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The postsingular set

$$¶(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}.$$

All inverse branches of all iterates are defined outside the postsingular set.

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Singular values and the Fatou set

- Attracting, superattracting, parabolic domains contain a singular value.
  (In fact, a critical point or an asymptotic path.)
- Boundaries of rotation domains are contained in the postsingular set.
- Limit functions in wandering domains are contained in the derived set of the postsingular set.
- Functions with finitely many singular values do not have wandering domains.
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The Fatou-Shishikura inequality

Theorem V.1.1 (Fatou-Shishikura inequality)

The number of nonrepelling cycles is bounded by the number of singular values.

Finite-type maps

Finite-type maps provide a natural generalization of rational functions, entire functions with finitely many singular values, meromorphic functions with finitely many singular values, . . .

Quadratic polynomials

$$z \mapsto \alpha z^2 + \beta z + \gamma, \quad \alpha \neq 0$$

- Only one singular value.
- Normalize near infinity, move critical point to zero:

$$f_c : z \mapsto z^2 + c.$$

- $K_c := K(f_c) = \{z \in \mathbb{C} : f^n(z) \not\to \infty\}$
- $J_c = \partial K_c$.

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Connectivity of $J(f_c)$

Theorem V.2.1

1. *If $c \in K_c$, then the Böttcher map ϕ_c extends to a conformal isomorphism*

$$\phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\mathbb{D}}.$$

In particular, K_c is connected.

2. *If $c \notin K_c$, then there is a unique maximal domain U such that ϕ_c extends to U and $\phi_c(U)$ is the outside of some closed disk.*
U contains c, and ∂U is a "figure-eight curve" symmetric around 0.

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The Mandelbrot set

$$\mathcal{M} := \{c \in \mathbb{C} : f_c^n(c) \not\to \infty\}.$$

Lemma V.2.2

1. *$\mathcal{M}$ is compact.*
2. *In fact,*

$$\mathcal{M} = \{c \in \mathbb{C} : |f_c^n(c)| \leq 2 \text{ for all } n \geq 0\}.$$

3. $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$.
4. *$\mathbb{C} \setminus \mathcal{M}$ is connected. In particular, every component of $\text{int}(\mathcal{M})$ is simply-connected.*

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Connectivity

Theorem V.2.3 (Douady-Hubbard)

The map

$$\Phi : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{\mathbb{D}}, \quad c \mapsto \phi_c(c)$$

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- External rays, puzzles, ...
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- Density of hyperbolicity, local connectivity.

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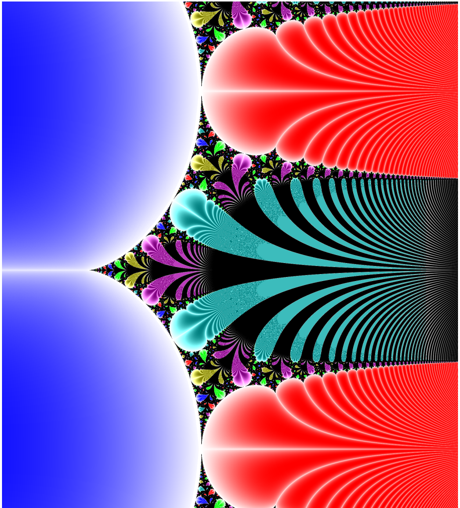
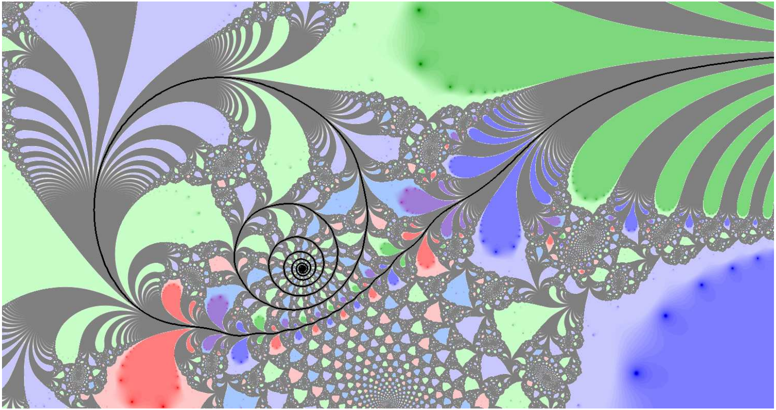
The exponential family

$$z \mapsto \exp(z) + \kappa$$



Dynamic rays



Parameter rays