An Introduction to Holomorphic Dynamics

IV. Local fixed point theory

L. Rempe

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This handout is created from the overhead slides used during lectures. Examples and proofs will be done on the board, and are not included.

IV.1 Introduction

Basic question

Question. Suppose z_0 is a fixed point of a holomorphic function f.

What is the *local dynamics* near z_0?

For example, we may ask

Question. Is f conformally conjugate to a linear map $z \mapsto \lambda z$ near z_0

Basic conjugacy results

Often in dynamics, conjugacy results come from *expansion*.

We will try to phras some conjugacy results in a way which emphasizes this idea.

IV.2 Hyperbolic fixed points

Linearization of attracting fixed points

IV.2.1 Theorem (Königs)**.** *Let* $\lambda \in \mathbb{D} \setminus \{0\}$*, and suppose that*

$$f(z) = \lambda z + O(z^2)$$

is holomorphic near 0.

Then there is a conformal map ϕ, defined in a neighborhood of 0, such that $\phi(0) = 0$ and

$$\phi(f(z)) = \lambda\phi(z).$$

Linearization of repelling fixed points

IV.2.2 Corollary. *Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$, and suppose that*

$$f(z) = \lambda z + O(z^2)$$

is holomorphic near 0.

Then there is a conformal map ϕ, defined in a neighborhood of 0, such that $\phi(0) = 0$ and

$$\phi(f(z)) = \lambda\phi(z).$$

IV.3 Böttcher coordinates

Böttcher coordinates

IV.3.1 Theorem. *If f has a superattracting fixed point z_0, then f is conjugate to $z \mapsto z^d$ near z_0.*

(Here d is the local degree of f near z_0.)

$$z \mapsto z^d + O(z^{d+1}).$$

Remark. The conjugacy is unique up to a choice of an $(m-1)$th root of unity.

IV.4 Fatou coordinates

Fatou coordinates

$$f(z_0) = z_0; \quad f'(z_0) = e^{2\pi i\theta}, \quad \theta \in \mathbb{Q}.$$

$$z \mapsto z + z^m + O(z^{m+1}).$$

Leau-Fatou flower

Fatou coordinates: f is conjugate to $z \mapsto z + 1$ in attracting petals.

IV.5 Irrational fixed points

Irrational fixed points

$$f(z_0) = z_0; \quad f'(z_0) = e^{2\pi i\theta}, \quad \theta \notin \mathbb{Q}.$$

- Linearizable: *Siegel disk*
- Not linearizable: *Cremer point*.

IV.6 Outlook

Outlook

1. The role of singular values.
2. Parameter spaces.
3. Hyperbolicity, structural stability, subhyperbolicity ...
4. Topological structure of the Julia set; escaping sets.
5. Geometry of the Julia set.
6. Measurable dynamics.
7. ...