An Introduction to Holomorphic Dynamics

III. Classification of Fatou Components

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Outline

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Reminder

X is either the plane, the Riemann sphere, or the punctured plane.

$f : X \to X$ is nonconstant and nonlinear.

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Periodic and wandering components

Definition III.1.1 (Periodic and Wandering Fatou Components)

Let U be a connected component of the Fatou set.

1. If $f(U) \subset U$, then we say U is an invariant Fatou component.
2. If $f^n(U) \subset U$, then we say U is periodic.
3. If $f^k(U)$ is contained in a periodic Fatou component V for some $k \geq 0$, then we say that U is eventually periodic.
4. Otherwise, U is called a wandering domain.

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Wandering domains

Theorem III.1.2 (Sullivan's "No Wandering Domains" Theorem)

Rational functions have no wandering domains.

Entire functions may have wandering domains:

$$f(z) = z + \sin(2\pi z).$$

Entire functions with finitely many singular values do not have wandering domains.

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Attracting and parabolic domains

In the following, let U be an invariant Fatou component of f.

Definition III.1.3 (Attracting and parabolic domains)

1. If $f^n|_U$ converges locally uniformly to some superattracting fixed point, then U is called a Böttcher domain.
2. If $f^n|_U$ converges locally uniformly to some attracting fixed point, then U is called an attracting domain.
3. If $f^n|_U$ converges locally uniformly to some fixed point $z_0 \in \partial U$ of f, then U is a parabolic domain.

Theorem III.1.4 (Parabolic points)

The boundary fixed point z_0 in (3) must have $f'(z_0) = 1$.

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Rotation domains

Definition III.1.5 (Rotation domains)

4. If U is simply connected, and $f|_U$ is conjugate to an irrational rotation, then U is a Siegel disk.
5. If U is doubly connected, and $f|_U$ is conjugate to an irrational rotation, then U is a Herman ring.

$$z \mapsto e^{2\pi i\theta} z(1-z) \quad \text{(suitable } \theta\text{)}.$$

Remark

Entire functions have no Herman rings.

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Baker domains

Definition III.1.6

6. If $f^n|_U$ converges locally uniformly to a point where f is not defined, then U is a **Baker domain**.

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$$z \mapsto z - 1 + \exp(z).$$

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Classification of invariant Fatou components

Theorem III.1.7 (Classification Theorem)

Every invariant Fatou component of f falls into one of the previously discussed categories.

A first version of the theorem

Theorem III.1.8 (Self-maps of hyperbolic domains)

Let $U \subset \mathbb{C}$ be open and connected, and assume U omits at least two points of $\mathbb{C}$.

Let $g : U \to U$ be holomorphic. Then exactly one of the following holds:

1. *The iterates g^n converge locally uniformly to a (super)-attracting fixed point;*
2. $\operatorname{dist}^{\#}(g^n(z), \partial U) \to 0$ *locally uniformly in U; or*
3. *$g : U \to U$ is a conformal isomorphism, and $g^{n_k} \to \operatorname{id}$ for some sequence of iterates of g.*

Remark

We usually think of g as the restriction of f to an invariant Fatou component U.

Then, in the second case, U must be a parabolic or Baker domain.

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The Riemann mapping theorem for Riemann surfaces

Theorem III.2.1 (Riemann mapping theorem)

Up to conformal isomorphism, every simply connected Riemann surface is either the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, or the unit disk $\mathbb{D}$.

Uniformization

Corollary III.2.2 (Uniformization)

Let U be a Riemann surface. Then there exists a holomorphic covering map

$$\pi : X \to U,$$

where $X \in \{\hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}\}$.

Uniformization of plane domains

Corollary III.2.3 (Plane domains)

Let $U \subset \hat{\mathbb{C}}$ omit at least three points. Then there is a holomorphic covering map $\pi : \mathbb{D} \to U$.

A deck transformation ϕ is a Möbius transformation of the disk such that

$$\pi \circ \phi = \pi.$$

The group Γ of deck transformations is discrete and fixed-point free, and

$$U \equiv \mathbb{D}/\Gamma.$$

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Lifts

Let U and $\pi : \mathbb{D} \to U$ as before, and let

$$f : U \to U$$

be holomorphic.

A lift $F : \mathbb{D} \to \mathbb{D}$ of f is a function satisfying

$$\pi \circ F = f \circ \pi.$$

If F and $\tilde{F}$ are such lifts, then there are deck transformations ϕ and ψ such that

$$\tilde{F} \circ \phi = F = \psi \circ \tilde{F}.$$

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Rotation domains

To finish the proof of the classification theorem, it remains to show:

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Suppose that $f : U \to U$ is such that some sequence f^{n_k} of iterates converges to the identity on U.

Then either U is simply or doubly connected, and f is conjugate to an irrational rotation, or $f^k|_U = \mathrm{id}$ for some k.

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