An Introduction to Holomorphic Dynamics

II. Properties of Julia and Fatou sets

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Outline

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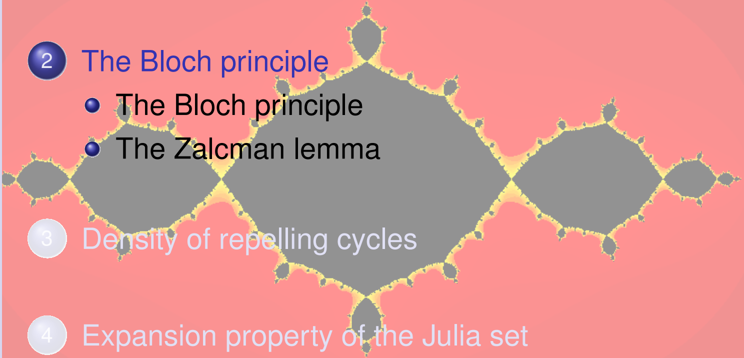

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Exceptional values

Definition II.1.1 (Exceptional value)

A value $z_0 \in \hat{\mathbb{C}}$ is called (Fatou) exceptional if the backward orbit

$$O^-(z_0) = \{w \in X : \exists n \geq 0, f^n(w) = z_0\}$$

is a finite set.

Example 1

- $f(z) = z^2$; $z_0 = 0$.
- $f(z) = \exp(z)$; $z_0 = 0$.

Lemma II.1.2 (Number of exceptional points)

f has at most two exceptional points in $\hat{\mathbb{C}}$.

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Remark

For rational functions of degree at least two, exceptional values are always in the Fatou set.

- A rational map with one exceptional value is conjugate to a polynomial.
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Density of backward orbits

We can now reformulate a property of the Julia set which we mentioned already in the previous lecture:

Lemma II.1.3 (Backward orbits)

If z_0 is not a Fatou exceptional value, then

$$J(f) \subset \overline{O^-(z_0)}.$$

If furthermore $z_0 \in J(f)$, then

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Liouville's Theorem

Recall that Liouville's Theorem states that a bounded entire function must be constant.

Compare this with the Removable Singularities Theorem, which says that an isolated singularity of a bounded holomorphic function is removable.

Also recall that any family of bounded entire functions, with a uniform bound, is normal.

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Theorems of Montel and Picard

Theorem II.2.1 (Picard)

Suppose f is meromorphic on a domain U, except at an isolated singularity $z_0 \in U$.
If f omits three values in the Riemann sphere (e.g., f never takes the values $0, 1$ and ∞), then z_0 is a removable singularity.

Theorem II.2.2 (Picard)

Any meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ which omits three values is constant.

Theorem II.2.3 (Montel)

A family of meromorphic functions which all omit the same three values is normal.

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The Bloch Principle

A property which implies that an entire (or meromorphic) function on the plane is constant should imply that a family of entire (or meromorphic) functions with this property is normal.

Of course, this heuristic principle isn't true as stated: for a trivial example, consider the property *f omits some collection of three points*.

(There are more interesting examples as well.)

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Zalcman's rescaling lemma

Larry Zalcman formulated a rescaling lemma which makes Bloch's heuristic principle explicit.

Theorem II.2.4 (Zalcman's Lemma)

The family f of meromorphic functions is not normal near a point z_0 if and only if:

There exists a sequence (f_n) in $\mathcal{F}$, a sequence $z_n \to z_0$, and a sequence of rescaling factors ρ_n with $\rho_n \to 0$ such that the functions

$$z \mapsto f_n(z_n + \rho_n z)$$

converge locally uniformly to a nonconstant meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}}$.

(Furthermore, f can be chosen with $f^{\#} \leq 1$ for all $z \in \mathbb{C}$.)

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It can not only be used to prove the equivalence of results for normal families and global analytic functions, but often also to prove such results themselves.

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Idea of the proof

- If $\mathcal{F}$ is not normal near z_0, then there is a sequence of points z_n and functions $f_n \in \mathcal{F}$ such that the spherical derivative tends to ∞ (by Marty's theorem).
- This gives us a sequence of rescalings of f_n with spherical derivative, say, bounded by 1.
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Proof of Montel's theorem from Picard's theorem

1. Let $\mathcal{F}$ be a family of functions on U, all of which omit the values $\{0, 1, \infty\}$.
2. If $\mathcal{F}$ is not normal, we can find a sequence of rescalings converging to a nonconstant entire function f.
3. The limit f must also omit $\{0, 1, \infty\}$ by Hurwitz's theorem.
4. This contradicts Picard's theorem.

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Periodic points

- $z \in \mathbb{C}$ is periodic if $f^n(z) = z$.
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Theorem II.3.1 (Density of repelling cycles)

Let $f : X \to X$ be nonlinear and nonconstant, as before, where $X \in \{\mathbb{C}, \hat{\mathbb{C}}, \mathbb{C}^*\}$.
Then repelling periodic points are dense in $J(f)$.

For rational functions, the usual proof uses the finiteness of nonrepelling cycles.

Baker's original proof for entire functions uses the five islands theorem.

We will give a proof using Zalcman's lemma, essentially due to Schwick (with simplifications due to Duval-Berteloot and Bargmann).

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Expansion property of the Julia set

As a consequence of the density of repelling periodic points, we can strengthen a number of properties of the Julia set.

Theorem II.4.1 (Expansion property)

Let $K \subset X$ be a compact set which does not contain any exceptional points.

If U is an open set with $U \cap J(f) \neq \emptyset$, then there is $n \geq 0$ with

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Existence of convergent subsequence

Lemma II.4.2

Let $z \in J(f)$. Then z has no neighborhood in which the sequence (f^n) has any uniformly convergent subsequence.