An Introduction to Holomorphic Dynamics

I. Introduction; Normal Families

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Liverpool, January 2008

Outline

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1. Introduction
   - Discrete dynamical systems
   - An example
2. Definition of Julia and Fatou sets
3. Normal families

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Discrete dynamical systems

General setting:

- X phase space;
- $f : X \to X$ function;
- $f^n = f \circ \cdots \circ f$ iterates of f;
- study behaviour of $f^n(x)$ as $n \to \infty$.

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Holomorphic dynamics

- X is a Riemann surface (i.e., a connected one-dimensional complex manifold);
- $f : X \to X$ is a holomorphic function.

Interesting behavior only for

$$X \in \{\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C} \setminus \{0\}, \mathbb{C}/\mathbb{Z}^2\}.$$

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Our setting

Standing Assumption I.1.1

X is either the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, or the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

$f : X \to X$ is a nonconstant holomorphic function which is not a conformal automorphism of X.

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Entire functions

Recall that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which is not a polynomial is called a transcendental entire function.

I.e.,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \neq 0$ for infinitely many k and the series converges for all $z \in \mathbb{C}$.

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Julia and Fatou sets

The phase space X can be partitioned into two fundamentally different sets:

- The Fatou set is the set where the dynamics is *regular*.
  This is an open set, and the possible types of behaviour are (fairly) well-understood.
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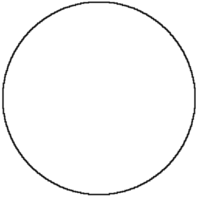
The simplest possible case

$$f(z) = z^2.$$



The quadratic family

$$f(z) = z^2 + c, \quad c \in \mathbb{C}.$$

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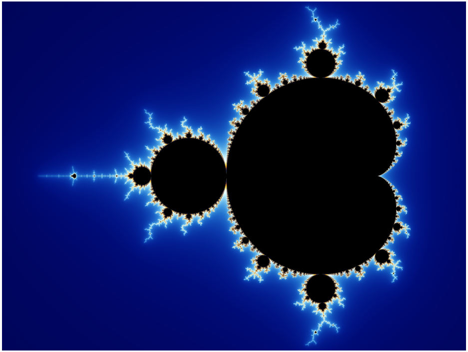

Equicontinuity

Recall that we want to define the Fatou set as the locus of stable behaviour.

This means that

small perturbations lead to small changes in long-term behaviour.

Definition I.2.1 (Equicontinuity)

Let A and B be metric spaces. A family $\mathcal{F}$ of functions from A to B is equicontinuous in a point $x_0 \in A$ if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall f \in \mathcal{F}\, \forall x \in A : \quad d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon.$$

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Fatou and Julia sets

Let X and $f : X \to X$ be as in our standing assumption.

Definition I.2.2 (Fatou set)

A point $z \in X$ belongs to the Fatou set $F(f)$ if there is a neighborhood U of z such that the family

$$\{f^n : n \in \mathbb{N}\}$$

is equicontinuous in every point of U (with respect to the *spherical* metric).

Definition I.2.3 (Julia set)

The Julia set of f is the complement of the Fatou set:

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Locally uniform convergence

Let f_n be a family of holomorphic (or meromorphic) functions defined on some open set U.

Recall that we say that (f_n) converges locally uniformly to a function f if the sequence converges uniformly on every compact subset of U.

(For example, the sequence $f_n(z) = z/n$ converges locally uniformly to $f(z) = 0$ on $\mathbb{C}$.)

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Results from Complex Analysis

Theorem I.3.1 (Schwarz Lemma)

Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$ (where $\mathbb{D}$ is the unit disk). Then

$$|f'(0)| \leq 1 \quad \textit{and} \quad |f(z)| \leq |z| \quad \textit{for all } z \in \mathbb{D},$$

with equality if and only if f is a rotation.

Theorem I.3.2 (Weierstraß theorem)

If $f_n \to f$ locally uniformly, where f_n and f are holomorphic functions defined on some open set $U \subset \mathbb{C}$, then $f_n' \to f'$ locally uniformly.

Theorem I.3.3 (Hurwitz theorem)

If $f_n \to f$ locally uniformly, as above, and $f_n(z) \neq 0$ for all z, then either $f \neq 0$ for all z, or f is constant.

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Normality

A family $\mathcal{F}$ of holomorphic or meromorphic functions on U is normal (on U) if every sequence of functions in $\mathcal{F}$ contains a locally uniformly convergent subsequence.

We say that $\mathcal{F}$ is normal in a point z if z has an open neighborhood on which $\mathcal{F}$ is normal.

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Theorem I.3.4 (Arzelà-Ascoli)

$\mathcal{F}$ is normal if and only if it is equicontinuous in every point of U.

(In particular, normality is a local property: $\mathcal{F}$ is normal if and only if it is normal in every point of U.)

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Marty's theorem

The spherical derivative of a meromorphic function f in z is

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The family $\mathcal{F}$ of meromorphic functions is normal if and only if the spherical derivatives in $\mathcal{F}$ are locally bounded.

(I.e., every $z_0 \in U$ has a neighborhood N such that $f^{\#}(z)$ is uniformly bounded in N, with the bound independent of $f \in \mathcal{F}$.)

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Two theorems of Montel

Theorem I.3.6 (Montel)

A uniformly bounded family of holomorphic functions is normal.

Theorem I.3.7 (Montel)

Let $a, b, c \in \hat{\mathbb{C}}$. Let $\mathcal{F}$ be a family of meromorphic functions on some open set U which omits the the three values a, b, c. (I.e., $f(z) \notin \{a, b, c\}$ for all $f \in \mathcal{F}$ and all z.) Then $\mathcal{F}$ is normal.

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Basic properties

Lemma I.3.8 (Basic properties of Julia and Fatou sets)

- *$F(f)$ is open; $J(f)$ is closed (in X).*
- *$F(f)$ and $J(f)$ are completely invariant; i.e.*

$$z \in F(f) \iff f(z) \in F(f).$$

- *Julia and Fatou sets are preserved under iteration. (That is, $F(f^n) = F(f)$, $J(f^n) = J(f)$.)*

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Properties of the Julia set

Theorem I.3.9 (Julia set infinite)

The Julia set $J(f)$ contains infinitely many points.

(Proof for entire functions: see course by Rippon and Stallard. Proof for rational functions: easy; see e.g. book by Milnor.)

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Consequences

Corollary I.3.10 (Backward orbits are dense)

For all points $z_0 \in \hat{\mathbb{C}}$ with at most three exceptions, the closure of the backward orbit

$$O^-(z_0) := \{w \in X : f^n(w) = z_0 \text{ for some } n \geq 0\}$$

contains the Julia set $J(f)$.

Corollary I.3.11 (Characterization of $J(f)$)

$J(f)$ is the smallest closed and backward invariant set containing at least three points.

Corollary I.3.12 (Julia sets with interior)

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$J(f)$ has no isolated points. In particular, $J(f)$ is unbounded.

Corollary I.3.14 (Dense orbits)

There exist (uncountably many) points $z \in J(f)$ such that the orbit

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Density of repelling periodic points

Definition I.3.15 (Periodic points)

A point $z \in X$ with $f^n(z) = z$ is called periodic.

(The smallest such n is the period of z.)

Such a periodic point is called

- attracting if $0 < |(f^n)'(z)| < 1$;
- superattracting if $|(f^n)'(z)| = 0$;
- repelling if $|(f^n)'(z)| > 1$;
- indifferent (or "neutral") if $|(f^n)'(z)| = 1$.

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Repelling periodic points are dense in the Julia set.

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