An Introduction to Holomorphic Dynamics

III. Classification of Fatou Components

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This handout is created from the overhead slides used during lectures. Examples and proofs will be done on the board, and are not included.

III.1 Classification of Fatou components (I)

III.1.1 Periodic Components and Wandering Domain

Reminder

X is either the plane, the Riemann sphere, or the punctured plane.

$f : X \to X$ is nonconstant and nonlinear.

$F(f)$ *Fatou set*, $J(f)$ *Julia set*

Periodic and wandering components

III.1.1 Definition (Periodic and Wandering Fatou Components). Let U be a *connected component* of the Fatou set.

1. If $f(U) \subset U$, then we say U is an *invariant Fatou component*.
2. If $f^n(U) \subset U$, then we say U is *periodic*.
3. If $f^k(U)$ is contained in a periodic Fatou component V for some $k \geq 0$, then we say that U is *eventually periodic*.
4. Otherwise, U is called a *wandering domain*.

Wandering domains

III.1.2 Theorem (Sullivan's "No Wandering Domains" Theorem)**.** *Rational functions have no wandering domains.*

Entire functions may have wandering domains:

$$f(z) = z + \sin(2\pi z).$$

Entire functions with finitely many *singular values* do not have wandering domains.

III.1.2 Classification of periodic Fatou components

Attracting and parabolic domains

In the following, let U be an invariant Fatou component of f.

III.1.3 Definition (Attracting and parabolic domains).

1. If $f^n|_U$ converges locally uniformly to some superattracting fixed point, then U is called a *Böttcher domain*.
2. If $f^n|_U$ converges locally uniformly to some attracting fixed point, then U is called an *attracting domain*.
3. If $f^n|_U$ converges locally uniformly to some fixed point $z_0 \in \partial U$ of f, then U is a *parabolic domain*.

III.1.4 Theorem (Parabolic points)**.** *The boundary fixed point z_0 in (3) must have* $f'(z_0) = 1$.

Rotation domains

III.1.5 Definition (Rotation domains).

4. If U is *simply connected*, and $f|_U$ is conjugate to an *irrational rotation*, then U is a *Siegel disk*.
5. If U is *doubly connected*, and $f|_U$ is conjugate to an *irrational rotation*, then U is a *Herman ring*.

$$z \mapsto e^{2\pi i\theta} z(1-z) \quad \text{(suitable } \theta\text{)}.$$

Remark. Entire functions have no Herman rings.

Baker domains

III.1.6 Definition. 6. If $f^n|_U$ converges locally uniformly to a point where f is not defined, then U is a *Baker domain*.

Remark. Rational functions have no Baker domains.

$$z \mapsto z - 1 + \exp(z).$$

Classification of invariant Fatou components

III.1.7 Theorem (Classification Theorem)**.** *Every invariant Fatou component of f falls into one of the previously discussed categories.*

III.1.3 A first classification

A first version of the theorem

III.1.8 Theorem (Self-maps of hyperbolic domains)**.** *Let $U \subset \mathbb{C}$ be open and connected, and assume U omits at least two points of $\mathbb{C}$.*

Let $g : U \to U$ be holomorphic. Then exactly one of the following holds:

1. *The iterates g^n converge locally uniformly to a (super)-attracting fixed point;*
2. $\operatorname{dist}^{\#}(g^n(z), \partial U) \to 0$ *locally uniformly in U; or*
3. *$g : U \to U$ is a conformal isomorphism, and $g^{n_k} \to \mathrm{id}$ for some sequence of iterates of g.*

Remark. We usually think of g as the restriction of f to an invariant Fatou component U.

Then, in the second case, U must be a *parabolic* or *Baker* domain.

III.2 Some hyperbolic geometry

The Riemann mapping theorem for Riemann surfaces

III.2.1 Theorem (Riemann mapping theorem)**.** *Up to conformal isomorphism, every simply connected Riemann surface is either the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, or the unit disk $\mathbb{D}$.*

Uniformization

III.2.2 Corollary (Uniformization). *Let U be a Riemann surface. Then there exists a holomorphic covering map*

$$\pi : X \to U,$$

where $X \in \{\hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}\}$.

Uniformization of plane domains

III.2.3 Corollary (Plane domains). *Let $U \subset \hat{\mathbb{C}}$ omit at least three points. Then there is a holomorphic covering map $\pi : \mathbb{D} \to U$.*

A deck transformation ϕ is a Möbius transformation of the disk such that

$$\pi \circ \phi = \pi.$$

The group Γ of deck transformations is discrete and fixed-point free, and

$$U \equiv \mathbb{D}/\Gamma.$$

Lifts

Let U and $\pi : \mathbb{D} \to U$ as before, and let

$$f : U \to U$$

be holomorphic.

A *lift* $F : \mathbb{D} \to \mathbb{D}$ of f is a function satisfying

$$\pi \circ F = f \circ \pi.$$

If F and $\tilde{F}$ are such lifts, then there are deck transformations ϕ and ψ such that

$$\tilde{F} \circ \phi = F = \psi \circ \tilde{F}.$$

III.3 Classification of Fatou components (II)

Rotation domains

To finish the proof of the classification theorem, it remains to show:

III.3.1 Theorem (Rotation domains). *Suppose that $f : U \to U$ is such that some sequence f^{n_k} of iterates converges to the identity on U.*

Then either U is simply or doubly connected, and f is conjugate to an irrational rotation, or $f^k|_U = \mathrm{id}$ for some k.